Journal of Pure and Applied Algebra 34 (1984) 57-115
North-Holland

ARBITRARY VS. REGULAR SEMIGROUPS*

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Communicated by J. Rhodes
Received 1 October 1983
Revised 21 October 1983

The notion of regularity for semigroups is studied, and it is shown that an unambiguous semigroup (i.e., whose $\mathscr{L}$ and $\mathscr{R}$ orders are respectively unions of disjoint trees) can be embedded in a regular semigroup with the same subgroups and the same ideal structure (except that a zero is added to the regular semigroup).

In a previous paper [1] it was shown that any semigroup is the homomorphic image of an unambiguous semigroup with the same groups and a similar ideal structure.

Together these two papers thus prove that an arbitrary semigroup divides a regular semigroup with a similar structure.

The resulting regular semigroup is finite (resp. torsion, or bounded torsion) if the given semigroup has that property.

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* This research was supported in part by US Air Force Grant AFOSR81-0238 and US Army Grant DAAG29-81-K-0136 through the Center for Mathematical System Theory, University of Florida, Gainesville, FL 32611, USA.

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1. Introduction

Definition. An element s of a semigroup S is *regular* iff there exists $x \in S$ such that $s = sxs$.

The semigroup S is said to be regular iff every element of S is regular. For undefined terms, see [5] and [7].

1.1. Meaning of regularity

The following intuitive interpretation of regularity of an element s was helpful in the constructions that will be given later.

Here we think of the semigroup S as a set of transformations acting on a set of states Q.

The transformation s is regular iff "s can be repeated, after it was applied a first time, and reproduces the same results." This is seen by interpreting the regularity property as follows:

$$\underbrace{s}_{\text{the transformation } s \text{ is applied (to states)}} \cdot \underbrace{x}_{\text{there exists something that } can \text{ be done (in } S\text{) such that}} \cdot \underbrace{s}_{\text{when } s \text{ is applied again}} = \underbrace{s}_{\text{we obtain the same effect as when } s \text{ was applied originally}} \tag{1.1}$$

In short we say that s is regular iff s is "*repeatable with the same results.*"

We shall prove a theorem, which reduces arbitrary semigroups to regular ones.

Semigroup expansions (treated in [1] and [2]) play a fundamental role; however this paper depends on these papers only through the *existence* of expansions having certain properties, and the reasoning refers only to those properties (– not to how they were obtained).

In particular we shall prove:

(a) For every semigroup S there exists a *regular* semigroup S_R such that $S \prec S_R$ (S divides S_R), and S_R has the same *subgroups* as S.

(b) A more precise statement is: For every semigroup S there exists a semigroup $\bar{S}$, a surmorphism $\varphi: \bar{S} \twoheadrightarrow S$, and a semigroup S_R such that: (i) $\bar{S} \le S_R$; (ii) S_R is regular; (iii) φ is injective when restricted to *subgroups* of $\bar{S}$; every non-trivial subgroup of S_R is D-equivalent to an isomorphic subgroup of $\bar{S}$.

Remarks. (1) Statement (a) follows, for *finite* semigroups, from the Allen–Rhodes synthesis theorem (see [9] and [3]) – as was observed by John Rhodes.

(2) See Section 2.5 for the complete statement of the theorem.

(3) The whole theorem grew out of an attempt to find a simpler proof of the Allen–Rhodes synthesis theorem -- which combines the Krohn–Rhodes and the Rees theorems (for finite semigroups). A relatively simple proof existed for regular finite semigroups (due to Stuart Margolis, J. Rhodes and D. Allen, Jr.) – and together with the "$S < S_R$"-theorem we obtain a new proof of the synthesis theorem. Also, the "$S < S_R$"-theorem extends the idea of a synthesis between the theory of regular semigroups and the global theory of arbitrary semigroups.

1.2. Examples of elementary embeddings of arbitrary semigroups in regular ones

1.2.1. Right regular representation

Let S be a semigroup and S^1 the monoid generated by S (i.e., $S^1 = S$ if S is a monoid; otherwise, $S^1 = S \cup \{1\}$ where 1 is a new element multiplied as $1 \cdot 1 = 1$, $1x = x1 = x$, $\forall x \in S$). Consider the semigroup $F(S^1 \to S^1)$ of all functions from S^1 into S^1, under composition.

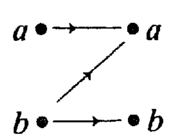
$F(S^1 \to S^1)$ is regular and $S \leq F(S^1 \to S^1)$ (using the embedding $s \hookrightarrow f_s$ where $(\forall x \in S^1)$: $(x)f_s = xs$ – see [5], [7] or [8, part II] for a more complete description).

The drawback of this embedding is that it does not preserve many properties of S; in fact $F(S^1 \to S^1)$ depends on S only by the *cardinality* of S^1.

1.2.2. Relations and their inverses

If Q is a set, define $B(Q)$ to be the semigroup of all binary relations on Q, under relational composition. Then $F(Q \to Q) \leq B(Q)$; hence by the right regular representation there exists Q such that $S \leq B(Q)$. For an element $s \in S \leq B(Q)$, denote the inverse relation by $s^{-1} \in B(Q)$.

Then $S \leq \langle S \cup \{s^{-1} \in B(Q) \mid s \in S\}\rangle_{B(Q)}$ (the subsemigroup of $B(Q)$, generated by $S \cup \{s^{-1} \mid s \in S\}$). One would guess that this semigroup is regular, since $s = ss^{-1}s$ and $s^{-1} = s^{-1}ss^{-1}$; moreover for relations $(r_1r_2)^{-1} = r_2^{-1}r_1^{-1}$ and $(r^{-1})^{-1} = r$. However $r = rr^{-1}r$ does *not* hold for arbitrary relations (but it does hold for functions and inverses of functions). *Example:* if $Q = \{a, b\}$ and r is defined by



then $(a)r = a$, but $(a)rr^{-1}r = (\{a, b\})r = \{a, b\}$. Neither $B(Q)$ nor $\langle S \cup \{s^{-1} \mid s \in S\}\rangle$ are regular in general (see Section 1.3, Fact 1.5).

But we shall see later in Section 2.4 that *if S is unambiguous,* then $\langle S \cup \{s^{-1} \mid s \in S\}\rangle$ has a homomorphic image which is *regular, contains S,* and whose subgroups divide the groups of S (at least in the finite case).

1.2.3. The following construction embeds an arbitrary semigroup S into a regular semigroup Reg(S). However many properties of S are lost when replacing it by Reg(S). Some of the lost properties (like inverse, orthodox, etc.) can be recovered if suitable relations (in terms of the generators) are imposed on Reg(S) (and then we obtain the constructions Inv(S), Orth(S), etc.).

The semigroups Reg(S), Inv(S), Orth(S),... will not be used as such in the rest of the paper; some of their properties are stated as conjectures, and further research is needed.

1.2.4. The semigroup Reg(S)

Let S be a semigroup and let $\bar{S}=\{\bar{s}\,|\,s\in S\}$ be a set disjoint from S and in one-to-one correspondence with S. Then Reg(S) is the semigroup presented by the set of generators $S\cup\bar{S}$ and the relations:

(1) $s_1 s_2 = s_3$ if $s_1\cdot s_2 = s_3$ (where $\cdot$ denotes multiplication in S).

(2) $\bar{s}_1\bar{s}_2=\overline{s_2\cdot s_1}$.

(3) If w is a word over $S\cup\bar{S}$, then $w=w\bar{w}w$, where $\bar{w}$ is defined as follows: if $w=(x_1,\ldots,x_n)\in(S\cup\bar{S})^+$ then $\bar{w}=(\bar{x}_n,\ldots,\bar{x}_1)$; here $\bar{x}\in S\cup\bar{S}$ is defined by:

$$\bar{x}:=\begin{cases}\bar{s} \text{ if } x=s\in S,\\ s \text{ if } x=\bar{s}\in\bar{S}.\end{cases}$$

Remark. By the relations (2), $\bar{S}$ can be considered to be the reverse semigroup of the semigroup S.

Relations (1) and (2) together define the free product of S and its reverse $\bar{S}$. So Reg(S) is the free product of S and $\bar{S}$, with the relations (3) imposed on it.

Remark. Reg(S) is a generalization of the so-called "free $*$-regular semigroup over a set of generators".

Properties of Reg(S). (i) Reg(S) is regular (by the relations (3)).

(ii) Reg($\cdot$) is a *functor:* If $\varphi: S\to T$ is a morphism, then there exists a morphism Reg(φ): Reg(S)$\to$Reg(T) (defined in the obvious way) etc. If φ is surjective, then Reg(φ) is surjective.

Conjecture. $S\le$ Reg(S).

This is harder to prove than it seems at first sight, but probably not too hard. E.g. the following reasoning outlines a proof that: if $s\not\equiv_J t$ in S, then $s\neq t$ in Reg(S).

Indeed, if $s\not\equiv_J t$, then $t\notin\{x\in S\,|\,x\ge_J s\}^\circ$ (Rees quotient), or conversely $s\notin\{x\in S\,|\,x\ge_J t\}^\circ$. Consider now the Rees quotient morphism $\varphi: S\twoheadrightarrow\{x\ge_J s\}^\circ$, and its functorial image Reg(φ): Reg(S)$\twoheadrightarrow$Reg($\{x\,|\,x\ge_J s\}^\circ$). We claim that under Reg(φ): $s\to s$ ($\neq 0$) and $t\mapsto 0$. That $t\mapsto 0$ is clear; to show that in Reg($\{x\,|\,x\ge_J s\}^\circ$), $s\neq 0$, observe that when the relations (1), (2) and (3) are applied, s is factored; but factors of s are all $\ge_J s$ hence never 0 in $\{x\,|\,x\ge_J s\}^\circ$.

Conjecture. Reg(S) may be infinite if S is finite. In fact, if $a,b\in S$ are not comparable in the $\le_J$-order, then $(a\bar{b})^n\neq(a\bar{b})^m$ if $n\neq m$.

We shall not use Reg(S) itself in the main part of this paper, but a construction $(S)_{\rm reg}$ such that: $(S)_{\rm reg}$ is a homomorphic image of Reg(S) with a zero added; and $S \le (S)_{\rm reg}$ if S is 'unambiguous' (defined later).

1.2.5. The semigroup Inv(S)

Inv(S) is defined to be Reg(S) with the following relations added:

(4) $\qquad w\bar{w}\bar{w}w = \bar{w}ww\bar{w}$, for any word w over $S \cup \bar{S}$.

1.1. Fact. Inv(S) *is an inverse semigroup.*

Proof. Regularity follows from the relations (3). We must show that all idempotents of Inv(S) commute.

Let e be any idempotent of Inv(S); then $e = \bar{e}$, for

$$\begin{aligned} e &= e\bar{e}e && \text{by (3)} \\ &= e\bar{e}\bar{e}e && \text{since } \bar{e}^2 = \bar{e} \\ &= \bar{e}ee\bar{e} && \text{by (4)} \\ &= \bar{e}e\bar{e} && \text{since } e = e^2 \\ &= \bar{e} && \text{by (3).} \end{aligned}$$

Also, the product of any two idempotents e, f of Inv(S) is an idempotent. Indeed let $e = e^2$, $f = f^2 \in \text{Inv}(S)$; then

$$\begin{aligned} ef &= ef \cdot \overline{ef} \cdot ef && \text{by (3)} \\ &= ef\bar{f}\bar{e}ef && \text{by (2)} \\ &= effeef && \text{since by the above: } e = \bar{e},\ f = \bar{f} \\ &= ef \cdot ef && \text{since } e = e^2,\ f = f^2 \\ &= (ef)^2. \end{aligned}$$

Now finally,

$$\begin{aligned} ef &= \overline{ef} && \text{since we proved that } ef \text{ is an idempotent} \\ &= \bar{f}\bar{e} && \text{by (2)} \\ &= fe && \text{since } e = \bar{e},\ f = \bar{f}. \qquad \square \end{aligned}$$

Conjecture. If in S the idempotents commute, then $S \le \text{Inv}(S)$.

Conjecture. Inv(S) can be infinite if S is finite.

1.2.6. The semigroup Orth(S)

Orth(S) is defined to be Reg(S) with the following relations added:

(4′) $(w_1\bar{w}_1\bar{w}_1w_1w_2\bar{w}_2\bar{w}_2w_2)^2 = w_1\bar{w}_1\bar{w}_1w_1w_2\bar{w}_2\bar{w}_2w_2$ for all words w_1, w_2 over $S\cup\bar{S}$.

1.2. Fact. Orth(S) *is an orthodox semigroup.*

Proof. Clearly Orth(S) is regular (by (3)). We must show that the product of any two idempotents e, f of orth(S) is an idempotent:

$$\begin{aligned} ef &= e\bar{e}ef\bar{f}f && \text{by (3)}\\ &= e\bar{e}\bar{e}ef\bar{f}\bar{f}f && \text{since } \bar{e}^2=\bar{e},\ \bar{f}^2=\bar{f}\\ &= (e\bar{e}\bar{e}ef\bar{f}\bar{f}f)^2 && \text{by (4′), letting } w_1=e \text{ and } w_2=f.\\ &= (ef)^2. && \square \end{aligned}$$

Remark. Axiom (4′) is a consequence of Fact 1.2 since $w_i\bar{w}_i$ and $\bar{w}_iw_i$ are idempotents.

Conjecture. If the idempotents of S form a subsemigroup, then $S \le$ Orth(S).

Conjecture. Orth(S) can be infinite if S is finite.

Remark. Other similar constructions can be devised, inspired from various semigroup properties. E.g.:

groups $G(S) = (S\cup\bar{S})^*/(s\bar{s}=\bar{s}s=1)$,

bicyclic $BC(S) = (S\cup\bar{S})^*/(\bar{s}s=1)$.

We shall not use these constructions in their general form in this paper and they need further research; they are generalizations of certain previously known constructions (that are 'free' in various ways) to arbitrary semigroups.

1.3. Counterexamples

Another notion that one could think of, but which does not exist, is the notion of the "*regular* subsemigroup generated by an arbitrary subsemigroup of a regular semigroup." I.e., if $S \le T$ and T is regular one could consider $\bigcap\{R/S \le R \le T,\ R$ regular$\}$; this subsemigroup exists, but it might not be regular.

1.3. Fact. *The intersection of two regular subsemigroups of a regular semigroup can be non-regular.*

Proof. Consider the regular semigroups S, and S_1, $S_2 \le S$ defined by

$S = \mathcal{M}^0(\{1,2\}\times\{e=e^2\}\times\{1,2,3\})$, $S_1 = \mathcal{M}^0(\{1,2\}\times\{e\}\times\{1,2\})$,

○	1	2	3
1	0	e	e
2	e	0	0

○	1	2
1	0	e
2	e	0

$S_2 = \mathcal{M}^0(\{1,2\} \times \{e\} \times \{1,3\})$

∘	1	3
1	0	e
2	e	0

(These are Rees-matrix semigroups, with matrices given above.) Then $S_1 \cap S_2 = \mathcal{M}^0(\{1,2\} \times \{e\} \times \{1\})$ is non-regular since it is given by

∘	1
1	0
2	e

1.4. Fact. *The intersection $\bigcap_{n\in\omega} R_n$ of a nested chain $R_1 \supset R_2 \supset \cdots \supset R_n \supset \cdots$ of regular semigroups can be empty, or non-empty and non-regular.*

Proof. Let $R_1 = \mathcal{M}^0(\{1,2\} \times \{e=e^2\} \times \omega + 1)$, i.e., R_1 is given by

∘	1	2	3	...	n	$\to\omega$	ω
1	e	e	e	...	e	e	0
2	0	0	0	...	0	0	e

and let $R_n = \mathcal{M}^0(\{1,2\} \times \{e=e^2\} \times \{n, n+1, \ldots, \omega\})$ (for $n \in \omega$), given by

∘	n	$n+1$	...	ω
1	e	e	e	0
2	0	0	0	e

also define

$R_\omega =$

∘
0
e

Clearly all semigroups R_n with $n \in \omega$ are regular, but $\bigcap_{n\in\omega} R_n = R_\omega$, and R_ω is not regular.

An example of a chain of regular semigroups whose intersection is empty is $R_1 = (\mathbb{N}, \max)$, $R_n = (\{x \in \mathbb{N} \mid x \geq n\}, \max\}$. □

Question. Can every semigroup S be *embedded* ($\leq$) in a regular semigroup which has thesame subgroups (or the same divisors) as S?

Guessed answer: No; there even exist combinatorial semigroups which can not be *embedded* in a regular combinatorial semigroup.

1.5. Fact. *Let K be any cardinal with $K \geq 4$, and let $B(K)$ be the semigroup of all binary relations on a set of K elements (under relational composition). Then $B(K)$ is non-regular.*

Proof. $B(K)$ can be described faithfully by $K\times K$ Boolean matrices – with entries in the semiring $(\{0,1\};+,\cdot)$, with $0+0=0$, $1+0=0+1=1+1=1$, and $1\cdot 1=1$, $1\cdot 0=0\cdot 1=0\cdot 0=0$.

Consider the element $x\in B(K)$ given by the matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 & \\ 0 & 1 & 1 & 0 & \mathbf{0} \\ 1 & 0 & 1 & 0 & \\ 1 & 0 & 0 & 0 & \\ & & \mathbf{0} & & \mathbf{0} \end{bmatrix}.$$

We claim that x is non-regular, i.e., $\forall y\in B(K)$: $x\neq xyx$. Let

$$y=\begin{bmatrix} a & b & c & d & * & \cdots \\ e & f & g & h & * & \cdots \\ i & j & k & l & * & \cdots \\ * & * & * & * & * & \cdots \\ \cdot & \cdot & \cdot & & & \cdots \\ \cdot & \cdot & \cdot & & & \cdots \end{bmatrix},$$

and assume $x=xyx$. Then

$$xyx=\begin{bmatrix} a+e & b+f & c+g & d+h & \cdots \\ e+i & f+j & g+k & h+l & \cdots \\ a+i & b+j & c+k & d+l & \cdots \\ a & b & c & d & \cdots \\ & & \mathbf{0} & & \mathbf{0} \end{bmatrix}\cdot\begin{bmatrix} 1 & 1 & 0 & 0 & \\ 0 & 1 & 1 & 0 & \mathbf{0} \\ 1 & 0 & 1 & 0 & \\ 1 & 0 & 0 & 0 & \\ & & \mathbf{0} & & \mathbf{0} \end{bmatrix}$$

$$=x=\begin{bmatrix} 1 & 1 & 0 & 0 & \\ 0 & 1 & 1 & 0 & \mathbf{0} \\ 1 & 0 & 1 & 0 & \\ 1 & 0 & 0 & 0 & \\ & & \mathbf{0} & & \mathbf{0} \end{bmatrix}$$

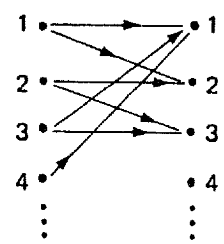
$$\Rightarrow\begin{cases} \text{(row 2)}\times\text{(column 1)}: e+i+g+k+h+l=0 \\ \quad\Rightarrow\quad e=i=g=k=h=l=0, \\ \text{(row 1)}\times\text{(column 3)}: b+f+c+g=0 \\ \quad\Rightarrow\quad b=f=c=g=0, \\ \text{(row 3)}\times\text{(column 2)}: a+i+b+j=0 \\ \quad\Rightarrow\quad a=i=b=j=0. \end{cases}$$

But also (row 1) × (column 2): $a+e+b+f=1$. This however is *impossible*, since we just obtained that $a=e=b=f=0$. □

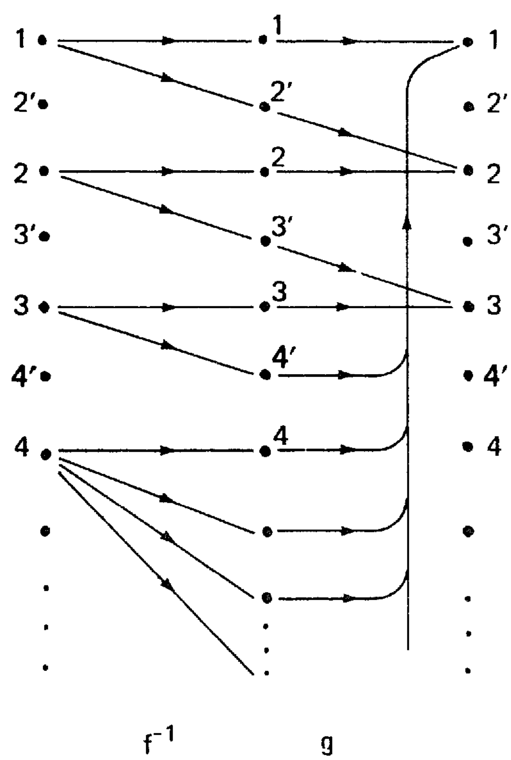
The relation

$$x=\begin{bmatrix} 1 & 1 & 0 & 0 & \\ 0 & 1 & 1 & 0 & \mathbf{0} \\ 1 & 0 & 1 & 0 & \\ 1 & 0 & 0 & 0 & \\ & \mathbf{0} & & & \mathbf{0} \end{bmatrix}$$

can be written as $x=f^{-1}\cdot g$, where f and g are *functions* (acting on the right). Indeed, the graph of x can be represented as:



which is equal to



assuming $Q=\{1,1',2,2',\ldots\}$.

Consider the semigroup $S=\langle f,g\rangle_{B(Q)}$. *Then the semigroup* $\langle S\cup\{s^{-1}\mid s\in S\}\rangle_{B(Q)}\subseteq B(Q)$ *is not regular* (*since it contains the above relation* x, *which is non-regular in* $B(Q)$).

2. Embedding an unambiguous semigroup in a regular semigroup

2.1. Results from [1]

Definition. A semigroup S has *unambiguous R-order* iff $(\forall x, y, z \in S)$: $y \geq_{\mathcal{R}} x$ and $z \geq_{\mathcal{R}} x$, implies y and z are R-comparable (the same definition can be made for the L-order).

A semigroup is *unambiguous* if both its R- and its L-order are unambiguous.

Definition. A semigroup S has $h_{\mathcal{R}}$ ('Dedekind height property' for the R-order) iff for any $x \in S$ there exists a bound (depending only on x) on the length of all $>_{\mathcal{R}}$-chains ascending from x (the same definition can be made for L).

Remark. A semigroup has $h_{\mathcal{R}}$ and unambiguous R-order iff the Hasse diagram of the $>_{\mathcal{R}}$-relation on $S/\equiv_{\mathcal{R}}$ is a union of disjoint rooted trees – so for every vertex there is a unique dense path to a root; moreover this dense path to the root is finite.

Definition. The semigroup S is *finite-J-above* iff $(\forall s \in S)$: the set $J(\geq s) = \{x \in S \mid x \geq_{J} s\}$ is finite.

Definition. The semigroup $S_s = \{x \in S \mid sx = s\}$ is called the *right-stabilizer* of s in S.

See [1] or [8, part II] for more details on the above definitions.

Definition. (Properties of surmorphisms). Let $\varphi : S \twoheadrightarrow T$ be a surmorphism of semigroups.

φ is *H-injective* iff the restriction of φ to any H-class of S is injective.

φ is *cyclic-injective* iff the restriction of φ to any cyclic subsemigroup of S is injective.

φ *preserves idempotents* iff for any idempotent $e \in T$, $(e)\varphi^{-1}$ consists only of idempotents (equivalently the inverse image of a band is a band).

φ *preserves groups* (in the weak sense) iff for any group $G \leq T$ there is a group $G' \subseteq (G)\varphi^{-1} \subseteq S$ such that $G = (G')\varphi$.

φ *preserves torsion-identities* iff $(\forall t \in T)$: t satisfies $t^{n+k} = t^n$ and $(s)\varphi = t \Rightarrow s$ satisfies $s^{n+k} = s^n$.

φ is D^* iff the inverse image of any regular D-class of T is a unique regular D-class of S.

φ is *strongly* J^* iff the inverse image of a set of J-equivalent regular elements of T is regular and is all contained in one J-class of S.

See [1] for more details on these definitions and the following theorem.

2.1. Theorem. *For any semigroup S, generated by a subset A, there exists a semigroup $\hat{S}^{\,\cdot}_A$, generated by a subset of cardinality $|A|$ (and also denoted by A),* *and a surmorphism $\eta : \hat{S}_A^+ \twoheadrightarrow S$ which is injective on A; the following properties hold for $\hat{S}_A^+$ and η:*

(1) *$\hat{S}_A^+$ is unambiguous and has* $\mathrm{h}_{\mathscr{L}}$ *and* $\mathrm{h}_{\mathscr{R}}$.

(2) *Non-regular H-classes of $\hat{S}_A^+$ are singletons.*

(3) *Left and right stabilizers in $\hat{S}_A^+$ are aperiodic.*

(4) *$\hat{S}_A^+$ is finite if S is finite; if S is infinite, then $\hat{S}_A^+$ has the same cardinality as S; if S is finite-J-above, then so is $\hat{S}_A^+$.*

(5) *η is H-injective and cyclic-injective.*

(6) *η is D* and strongly J*.*

(7) *η preserves groups (weakly) and torsion-identities.*

This theorem means that in global semigroup theory we can replace any semigroup S by $\hat{S}_A^+$, and obtain properties (1) to (4) – provided the preservation properties (5)–(7) of are good enough for our applications.

We shall show next that an unambiguous semigroup whose non-regular H-classes are singletons (e.g., $\hat{S}_A^+$ for any S) can be embedded in a regular semigroup having the same subgroups as the given semigroup.

2.2. *The construction* $(S)_{\mathrm{reg}}$

Let S be an *unambiguous* semigroup and let $\bar{S} = \{\bar{s} \mid s \in S\}$ be a set that is disjoint from S. Let 0 be an additional element which is neither in S nor in $\bar{S}$.

Let $(S)_{\mathrm{reg}}$ be the semigroup defined by the generators $S \cup \bar{S} \cup \{0\}$ and the following axioms:

(1) $s_1 s_2 = s_3$ if $s_1 \cdot s_2 = s_3$ in S (where $\cdot$ denotes the multiplication of S).

(2) $\bar{s}_1 \bar{s}_2 = \bar{s}_3$ if $s_2 \cdot s_1 = s_3$ in S.

(3) $0\,0 = 0$.

Remark. The semigroup generated by $S \cup \bar{S} \cup \{0\}$ and satisfying (1), (2), (3), is the *free product* of the semigroup $S, \bar{S}$ (considered to be the reverse semigroup of S), and $\{0\}$ (the one-element semigroup). On this free product we add the following axioms:

(4) $0s = s0 = 0 = 0\bar{s} = \bar{s}0$, for any $s \in S$ (i.e., 0 acts as a zero).

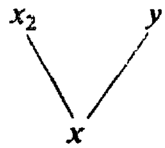
(5) $s\bar{s}s = s$ and $\bar{s}s\bar{s} = \bar{s}$, for any $s \in S$ (i.e., s and $\bar{s}$ are inverses).

(6L) $s_1 \bar{s}_2 = 0$ if $s_1 \not\gtreqless_{\mathscr{L}} s_2$ (where $\not\gtreqless_{\mathscr{L}}$ denotes incomparability in the L-order of S).

(6R) $\bar{s}_1 s_2 = 0$ if $s_1 \not\gtreqless_{\mathscr{R}} s_2$ (where $\not\gtreqless_{\mathscr{R}}$ denotes incomparability in the R-order of S).

(See [5, Vol. 2], and [6] for the free product of semigroups and related topics.)

Remark. Unambiguity of S is required for the following reason: Suppose $x <_{\mathscr{L}} y$, so $\exists a \in S$: $x = ay$; therefore $x = x\bar{x}x = x\overline{ay}x = x\bar{y}\bar{a}x$. But we *could* have also $x = x_1 \cdot x_2$ with $x_2 \not\gtreqless_{\mathscr{L}} y$; so $x_2\bar{y} = 0$, thus $x\bar{y} = x_1 x_2 \bar{y} = x_1 0 = 0$, which implies $\underline{x = x\bar{y}\bar{a}x = 0\bar{a}x = 0}$. This we want to avoid since we want $S \leq (S)_{\mathrm{reg}}$. However, in this case the L-order is ambiguous, since



(i.e., $x \leq_{\mathscr{L}} x_2$; $x \leq_{\mathscr{L}} y$, $x_2 \not\lessgtr_{\mathscr{L}} y$).

Similar remarks apply to the R-order.

Remark (*Intuitive idea* of the construction). Since we want to embed S in a regular semigroup we have to introduce regular inverses – hence we have relation (5) (here we actually introduce a new inverse for every element of S; later we shall discuss the possibility of introducing new inverses only for non-regular elements); we want these new inverses to be regular, which follows from relations (2) and (5). Relation (1) is needed if we want S to be embedded in the new semigroup.

Axioms ($6R, L$) are critical ones; as we shall prove very soon, their effect is to make products of old elements $s_k \in S$ and new elements $\bar{s}_i \in \bar{S}$ regular. Recall (1.1) where we argued that regularity means "repeatability with the same results"; intuitively, one way to obtain repeatability of a transformation s is "to go back into the past" up to the moment then s was applied first; call $\bar{s}$ this action of going into the past before s; so now we have the product $s\bar{s}$ (in the group case, the backwards movement $\bar{s}$ erases s; in the semigroup case $\bar{s}$ is "superimposed" on s). However, going back into the past is related to the L-order: if $s = s_1 s_2 \cdots s_{n-1} s_n$, then the last action was s_n, the previous last action was $s_{n-1} s_n$; before that the last action was $s_{n-2} s_{n-1} s_n$ etc.; of course $s_1 s_2 \cdots s_n \leq_{\mathscr{L}} \cdots \leq_{\mathscr{L}} s_{n-1} s_n \leq_{\mathscr{L}} s_n$.

Unambiguity of the L-order means here that there is a *unique path* back into the past (although we do not know uniquely how *far* back in the past the last action occurred). Axiom (6L) now means that if we apply s_1 and we than go back into the past by $\bar{s}_2$, we make $s_1 \bar{s}_2$ *undefined* (this is what 0 means) if s_2 does not lie on the path into the past on which s_1 is (i.e., if we try to *go into a past that could not have happened*).

Dually, the R-order can be interpreted as forward movement in time; axiom (6R) means, that we went into the past by the amount $\bar{s}_1$, and after that we move forward in time by the amount s_2. If however s_2 does not lie on the forward path that was used by $\bar{s}_1$ to go backwards in time, then we make $\bar{s}_1 s_2$ undefined ($=0$).

In both cases: we do not go backwards on a path that is incompatible with previous forward movements, and we do not go forward on a path that is incompatible with earlier backwards movements. This, intuitively, should avoid the introduction of new groups (cycles).

All this appears more clearly in the following.

2.3. *Normal form of elements of $(S)_{\text{reg}}$, and regularity*

2.2. Fact. *Every element w of $(S)_{\text{reg}}$ is either 0 or can be written (in a not neces-*

sarily unique way) in one of the following two normal forms:

(1) $$r=(s_1)\bar{t}_1 s_2 \bar{t}_2 \cdots \bar{t}_{k-1} s_k \bar{t}_k \cdots s_{n-1} \bar{t}_{n-1}(s_n)$$

(elements in parentheses may or may not be present in the product) with

$$(s_1 >_{\mathcal{L}})\ t_1 >_{\mathcal{R}} s_2 >_{\mathcal{L}} t_2 >_{\mathcal{R}} \cdots >_{\mathcal{L}} \underbrace{t_{k-1} \geq_{\mathcal{R}} s_k \leq_{\mathcal{L}} t_k}_{} <_{\mathcal{R}} \cdots <_{\mathcal{R}} s_{n-1} <_{\mathcal{L}} t_{n-1}\ (<_{\mathcal{R}} s_n)$$

not *both* $\equiv$ (i.e., $\cdot > s_k < \cdot$ or $\cdot \geq s_k < \cdot$ or $\cdot > s_k \leq \cdot$).

The element s_k is called the center of the normal form. The subwords that are left, resp. right, of the center, are called the left-, resp. right side.

(2) $$r=(s_1)t_1 s_2 t_2 \cdots s_k \bar{t}_k s_{k+1} \cdots s_{n-1} \bar{t}_{n-1}(s_n)$$

(again, elements in parentheses may or may not be present) with

$$(s_1 >_{\mathcal{L}})\ t_1 >_{\mathcal{R}} s_2 >_{\mathcal{L}} t_2 >_{\mathcal{R}} \cdots >_{\mathcal{R}} s_k >_{\mathcal{L}} t_k <_{\mathcal{R}} s_{k+1} <_{\mathcal{L}} \cdots <_{\mathcal{R}} s_{n-1} <_{\mathcal{L}} t_{n-1}\ (<_{\mathcal{R}} s_n).$$

The element $\bar{t}_k$ is called the center of this normal form.

Remark. Strictly speaking, the normal form is not the element $r \in (S)_{\text{reg}}$, but the sequence $w=((s_1,\ )\bar{t}_1, \ldots, \bar{t}_{n-1}(\ , s_n)) \in (S \cup \bar{S})^+$, with components alternately in S and $\bar{S}$, and satisfying the L- and R-orderings given in (1) and (2). (*Notation:* A^+ is the free semigroup over the set of generators A.)

It is convenient to use the following graphical representation of normal forms (which is related to the remarks on forward and backward movement made earlier). The normal form $s_1 \bar{t}_1 s_2 \bar{t}_2 \cdots \bar{t}_{k-1} s_k \bar{t}_k \cdots s_{n-1} \bar{t}_{n-1} s_n$ with

$$s_1 >_{\mathcal{L}} t_1 >_{\mathcal{R}} s_2 >_{\mathcal{L}} t_2 >_{\mathcal{R}} \cdots >_{\mathcal{L}} t_{k-1} \geq_{\mathcal{R}} s_k \leq_{\mathcal{L}} t_k <_{\mathcal{R}} \cdots <_{\mathcal{R}} s_{n-1} <_{\mathcal{L}} t_{n-1} <_{\mathcal{R}} s_n$$

will be drawn as in Fig. 1.

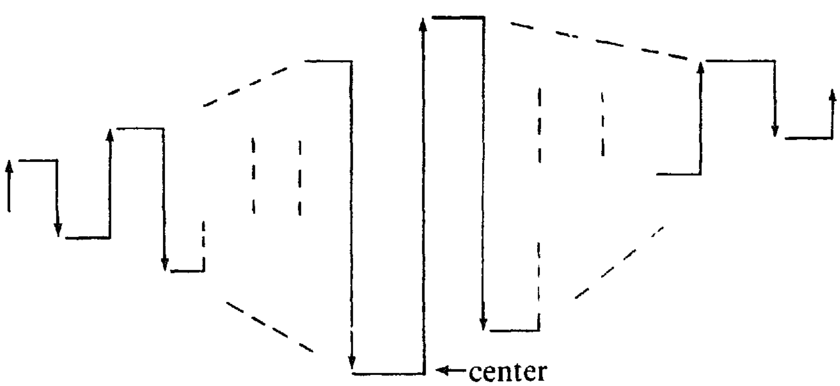
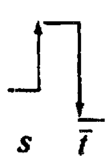
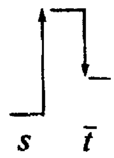


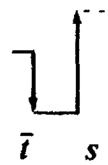
Fig. 1.

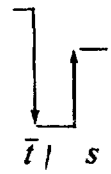
Here, elements $s_i \in S$ are represented by upward arrows (cf. forward movement) and elements $\bar{t}_j \in \bar{S}$ are represented by downward arrows (backward movement). The relative length of arrows represents the L-resp. R-depth of the components:

in 

we have $s >_{\mathcal{L}} t$; in 

we have $s <_{\mathcal{L}} t$;

in 

we have $t >_{\mathcal{R}} s$; in 

we have $t <_{\mathcal{R}} s$.

Similarly, the normal form $s_1\bar{t}_1 s_2\bar{t}_2 \cdots s_k\bar{t}_k s_{k+1} \cdots s_{n-1}\bar{t}_{n-1}s_n$ with

$$s_1 >_{\mathcal{L}} t_1 >_{\mathcal{R}} s_2 >_{\mathcal{L}} t_2 >_{\mathcal{R}} \cdots >_{\mathcal{R}} s_k >_{\mathcal{L}} t_k <_{\mathcal{R}} s_{k+1} <_{\mathcal{L}} \cdots <_{\mathcal{R}} s_{n-1} <_{\mathcal{L}} t_{n-1} <_{\mathcal{R}} s_n$$

is drawn as in Fig. 2.

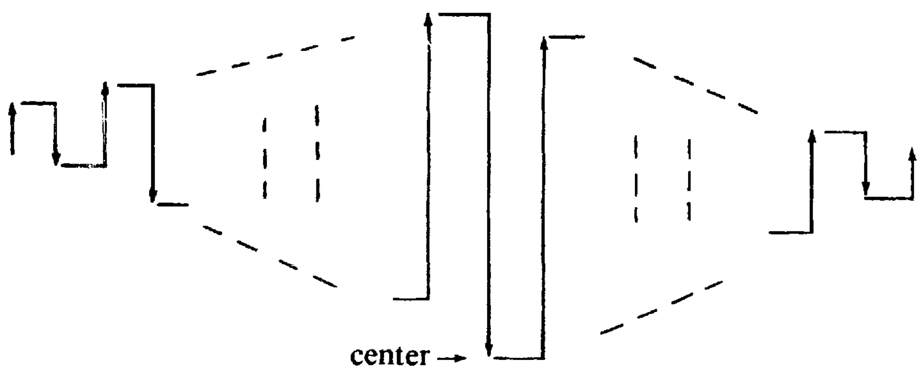


Fig. 2.

Proof of (2.2). We start out with any word in $(S\cup\bar{S})^+$. By axioms (1) and (2), this word is equivalent to one in which the components are alternately in S and $\bar{S}$, i.e., now every subsegment of length 2 has either the form $\bar{x}y$ or $x\bar{y}$. If for the subsegment $\bar{x}y$, we have $x \not\gtreqless_{\mathcal{R}} y$ then $\bar{x}y = 0$ (by axiom (6R)), so the whole word is equivalent to 0 (by axiom (4)). Similarly, if for the subsegment $x\bar{y}$ we have $x \not\gtreqless_{\mathcal{L}} y$, then $x\bar{y} = 0$ and then the whole word will be equivalent to 0.

Let us assume now that adjacent components of the word w are *L-, resp. R-comparable*. We can prove then, by induction on the length of the word w $(\in (S\cup\bar{S})^+)$, that it is equivalent to a word in one of the above normal forms.

If the word w has length ≤ 2, then it is in normal form.

If the word w has length ≥ 3, then it contains a subsegment of the form $\bar{x}y\bar{z}$ with $x \gtreqless_{\mathcal{R}} y \gtreqless_{\mathcal{L}} z$, or a subsegment $x\bar{y}z$ with $z \gtreqless_{\mathcal{L}} y \gtreqless_{\mathcal{R}} z$.

Let us consider the case $\bar{x}y\bar{z}$; the comparability relations take one of the following forms:

$$x <_{\mathcal{R}} y <_{\mathcal{L}} z \quad \text{or} \quad x >_{\mathcal{R}} y >_{\mathcal{L}} z \quad \text{or} \quad \underbrace{x \ge_{\mathcal{R}} y \le_{\mathcal{L}} z}_{\text{not } both \,\equiv} \quad \text{or} \quad x \le_{\mathcal{R}} y \ge_{\mathcal{L}} z.$$

In the latter case we can reduce the length of the word w as follows: since there exist $u, v \in S^1$ with $x = yu$, $z = vy$, we can write the subsegment $\bar{x}y\bar{z}$ as $\overline{yu}y\overline{vy}$, which by axiom (2) is equivalent to $\bar{u}\bar{y}y\bar{y}\bar{v}$ (if u or $v = 1 \in S^1$ we do not write $\bar{u}$ or $\bar{v}$); this is equivalent (by axiom (5)) to $\bar{u}\bar{y}\bar{v}$ which is $\overline{vyu}$ (by axiom (2)); hence we have replaced the subsegment $\bar{x}y\bar{z}$ (of length 3) by the subsegment $\overline{vyu}$ of length 1.

The case $x\bar{y}z$ is dealt with similarly: the comparability relations are

$$x <_{\mathcal{L}} y <_{\mathcal{R}} z \quad \text{or} \quad x >_{\mathcal{L}} y >_{\mathcal{R}} z \quad \text{or} \quad \underbrace{x \geq_{\mathcal{L}} y \leq_{\mathcal{R}} z}_{\text{not } both\, \equiv} \quad \text{or} \quad x \leq_{\mathcal{L}} y \geq_{\mathcal{R}} z.$$

In the latter case we can reduce the length of the word w.

Inductively, we obtain that the word w is equivalent to a word in which all adjacent components are comparable (R or L as given by axiom (6)) but such that the comparability relations for three adjacent components always take the form $x<y<z$ or $x>y>z$ or $x \geq y \leq z$ (not both $\equiv$) – where R and L alternate. The case $x \leq y \geq z$ does not occur anymore.

It is now easy to see that the configuration $x_1 < y_1$ cannot occur left of the configuration $x_2 > y_2$ (otherwise, since the word is finite, at some point there is a transition from $<<<\cdots$ to $\cdots>>>$, where we have then $\cdot\leq\cdot\geq\cdot$; this contradicts the assumption that this configuration has been eliminated). Therefore the orderings in the word take the shape $>>\cdots>\geq\leq<\cdots<<$; at the center we have $\cdot\geq\cdot\leq\cdot$ (not both $\equiv$, since $\cdot\equiv\cdot\equiv\cdot$ is an instance of $\cdot\leq\cdot\geq\cdot$).

This proves the fact. □

2.3. Corollary. *If S is finite, then $(S)_{\mathrm{reg}}$ is finite.*

2.4. Fact. *The semigroup $(S)_{\mathrm{reg}}$ is regular.*

Proof. We shall prove that if $w = (s_1)\bar{t}_1 s_2 \bar{t}_2 \cdots \bar{t}_{k-1} s_k \bar{t}_k \cdots s_{n-1} \bar{t}_{n-1}(s_n)$ with

$$(s_1 >_{\mathcal{L}})\ t_1 >_{\mathcal{R}} s_2 >_{\mathcal{L}} t_2 >_{\mathcal{R}} \cdots >_{\mathcal{L}} t_{k-1} \geq_{\mathcal{R}} s_k \leq_{\mathcal{L}} t_k <_{\mathcal{R}} \cdots <_{\mathcal{R}} s_{n-1} <_{\mathcal{L}} t_{n-1}\ (<_{\mathcal{R}} s_n),$$

then $w' = (\bar{s}_n) t_{n-1} \bar{s}_{n-1} \cdots t_k \bar{s}_k t_{k-1} \cdots t_2 \bar{s}_2 t_1 (\bar{s}_1)$ with

$$(s_n >_{\mathcal{R}})\ t_{n-1} >_{\mathcal{L}} s_{n-1} >_{\mathcal{R}} \cdots >_{\mathcal{R}} t_k \geq_{\mathcal{L}} s_k \leq_{\mathcal{R}} t_{k-1} <_{\mathcal{L}} \cdots <_{\mathcal{R}} t_2 <_{\mathcal{L}} s_2 <_{\mathcal{R}} t_1\ (<_{\mathcal{L}} s_1),$$

is an inverse for w (i.e., $w = ww'w$).

This will be proved by induction on n, the length of the shortest normal form representation that exists for the element in $(S)_{\mathrm{reg}}$.

First, those elements having a normal form representation of length 1 satisfy our claim (this is the content of axiom (5): $s = s\bar{s}s$ and $\bar{s} = \bar{s}s\bar{s}$).

Assume now that elements having a normal form of length shorter than the length of w (and center in S) satisfy our claim. Then

$$ww'w$$
$$= [(s_1)\bar{t}_1 s_2 \bar{t}_2 \cdots s_{n-1} \bar{t}_{n-1}(s_n)][(\bar{s}_n) t_{n-1} \bar{s}_{n-1} \cdots t_2 \bar{s}_2 t_1 (\bar{s}_1)][(s_1)\bar{t}_1 s_2 \bar{t}_2 \cdots s_{n-1} \bar{t}_{n-1}(s_n)].$$

Since $t_{n-1} <_{\mathscr{R}} s_n$, $s_1 <_{\mathscr{L}} t_1$, there exist $a, b \in S$ such that $t_{n-1} = s_n a$, $s_1 = bt_1$; so

$$\begin{aligned} ww'w &= [(s_1)\bar{t}_1 s_2 \cdots s_{n-1}\bar{t}_{n-1}(s_n)][(\bar{s}_n)\cdot s_n a \cdot \bar{s}_{n-1} \cdots t_2 \bar{s}_2 \cdot bs_1 \cdot (\bar{s}_1)][(s_1)\bar{t}_1 s_2 \cdots s_{n-1}\bar{t}_{n-1}(s_n)] \\ &= (s_1)\bar{t}_1 s_2 \cdots s_{n-1}\bar{t}_{n-1} s_n a \bar{s}_{n-1} \cdots t_2 \bar{s}_2 b s_1 \bar{t}_1 s_2 \cdots s_{n-1}\bar{t}_{n-1}(s_n) \\ &= (s_1)[\bar{t}_1 s_2 \cdots s_{n-1}\bar{t}_{n-1}][t_{n-1}\bar{s}_{n-1} \cdots t_2 \bar{s}_2 t_1][\bar{t}_1 s_2 \cdots s_{n-1}\bar{t}_{n-1}](s_n). \end{aligned}$$

Thus we have reduced the length of the normal form. We can continue as follows: since $t_{n-1} >_{\mathscr{L}} s_{n-1}$, $t_1 >_{\mathscr{R}} s_2$ there exist $c, d \in S$, with $s_{n-1} = ct_{n-1}$, $s_2 = t_1 d$; then

$$\begin{aligned} ww'w &= (s_1)[\bar{t}_1 s_2 \bar{t}_2 \cdots ct_{n-1} \cdot \bar{t}_{n-1}][t_{n-1}\bar{s}_{n-1} \cdots t_2 \bar{s}_2 t_1][\bar{t}_1 \cdot t_1 d \cdot \bar{t}_2 \cdots s_{n-1}\bar{t}_{n-1}](s_n) \\ &= (s_1)\bar{t}_1 s_2 \bar{t}_2 \cdots ct_{n-1}\bar{s}_{n-1} \cdots t_2 \bar{s}_2 t_1 d\bar{t}_2 \cdots s_{n-1}\bar{t}_{n-1}(s_n) \\ &= (s_1)\bar{t}_1 [s_2 \bar{t}_2 \cdots s_{n-1}][\bar{s}_{n-1} \cdots t_2 \bar{s}_2][s_2 \bar{t}_2 \cdots s_{n-1}]\bar{t}_{n-1}(s_n). \end{aligned}$$

Thus, inductively, we obtain $ww'w = w$.

The case of elements of $(S)_{\text{reg}}$ representable by a normal form with center in $\bar{S}$ is treated similarly: the inverse of $(s_1)\bar{t}_1 s_2 \bar{t}_2 \cdots s_k \bar{t}_k s_{k+1} \cdots s_{n-1}\bar{t}_{n-1}(s_n)$ with $(s_1 >_{\mathscr{L}})\, t_1 >_{\mathscr{R}} s_2 >_{\mathscr{L}} t_2 >_{\mathscr{R}} \cdots >_{\mathscr{R}} s_k >_{\mathscr{L}} t_k <_{\mathscr{R}} s_{k+1} <_{\mathscr{L}} \cdots (<_{\mathscr{R}} s_n)$ is

$$(\bar{s}_n)t_{n-1}\bar{s}_{n-1} \cdots \bar{s}_{k+1} t_k \bar{s}_k \cdots t_2 \bar{s}_2 t_1 (\bar{s}_1)$$

with

$$(s_n >_{\mathscr{R}})\, t_{n-1} >_{\mathscr{L}} s_{n-1} >_{\mathscr{R}} \cdots >_{\mathscr{L}} s_{k+1} >_{\mathscr{R}} t_k <_{\mathscr{L}} s_k <_{\mathscr{R}} \cdots <_{\mathscr{R}} t_2 <_{\mathscr{L}} s_2 <_{\mathscr{R}} t_1 \,(<_{\mathscr{L}} s_1).$$

Finally, the case of the element 0 is trivial since $0\,0\,0 = 0$ (axiom (3)). □

2.5. Fact. *The normal form representations of elements of* $(S)_{\text{reg}}$ *is usually not unique. The following holds:*

(a) *If* $as \equiv_{\mathscr{L}} s$, *then* $\overline{as} \cdot as = \bar{s}s$ *in* $(S)_{\text{reg}}$.

(b) $sb \equiv_{\mathscr{R}} s$, *then* $sb \cdot \overline{sb} = s\bar{s}$ *in* $(S)_{\text{reg}}$.

Proof of (a). ((b) is proved dually.)

$\overline{as}as = \overline{as} \cdot as\bar{s}s$	(since $s\bar{s}s$)
$= \overline{as} \cdot as \cdot \overline{cas} \cdot s$	(since $as \equiv_{\mathscr{L}} s$, $\exists c \in S^1$: $cas = s$)
$= \overline{as} \cdot as \cdot \overline{as} \cdot \bar{c} \cdot s$	(by axiom (2); $\bar{c}$ is dropped if $c = 1$)
$= \overline{as} \cdot \bar{c} \cdot s$	(by axiom (5))
$= \overline{cas} \cdot s$	(since $\overline{cas} = \overline{as} \cdot \bar{c}$, axiom (2))
$= \bar{s} \cdot s.$	

2.6. Corollary. (a) *If* $as \equiv_{\mathscr{L}} s$, *then* $(\forall t_1, t_2 \in S)$: $\overline{ast_1} \cdot ast_2 = \overline{st_1} \cdot st_2$.

(b) *If* $sb \equiv_{\mathscr{R}} s$, *then* $(\forall t_1, t_2 \in S)$: $t_1 sb \cdot \overline{t_2 sb} = t_1 s \cdot \overline{t_2 s}$.

Proof of (a). $\overline{ast_1} \cdot ast_2 = \bar{t}_1\, \overline{as} \cdot as\, t_2 = \bar{t}_1\, \bar{s}s\, t_2$ (by the fact). Now use axiom (2).

2.7. Corollary. *Elements of* $(S)_{\text{reg}}$ *that do not belong to* $S \cup \tilde{S} \cup \{0\}$ *do not have unique normal forms.*

We shall see in the appendix that the above fact is the "only source of non-uniqueness" of the representation by normal forms.

2.4. Relations and their inverses

Before we deal with the non-uniqueness of the representation of an element of $(S)_{\text{reg}}$ by normal forms, and prove the main properties of $(S)_{\text{reg}}$ (regularity, embedding of S, etc.), we revisit example (1.2.2) and see how it can be made to work (i.e., the group divisors of S are preserved, and we obtain regularity).

Recall the idea of 1.2.2: embed $S \le B(S^1)$ (the semigroup of all binary relations on the set S^1, under composition of relations). Within $B(S^1)$ consider the subsemigroup S_B which is generated by the set $S \cup \{s^{-1} \mid s \in S\}$ (where s^{-1} denotes the inverse relation of s). Then $S \le S_B$. This semigroup could be non-regular, and one can show that it may contain groups that do not divide S (i.e., that are not homomorphic image of a subsemigroup of S).

Let us now introduce the additional assumption that S is *unambiguous.*

2.8. Fact. *Assume S has unambiguous L-order. Then* $s_1 s_2^{-1} = 0$ *in* S_B *iff* $s_1 \not\gtreqless_{\mathcal{L}} s_2$ *in* S (*where* 0 *is the empty relation*).

Proof. We have: $s_1 s_2^{-1} \ne 0$ iff $(\exists x \in S^1)$: $(x)s_1 s_2^{-1} \ne \emptyset$ iff $(\exists x, u \in S^1)$: $u \in (x)s_1 s_2^{-1}$ iff $(\forall x, u \in S^1)$: $us_2 = xs_1$.

($\Rightarrow$) If for some $u, x \in S^1$: $us_2 = xs_1$, then s_2 and $s_1 \ge_{\mathcal{L}} us_2$ $(= xs_1)$. By unambiguity of the L-order of S, this implies $s_1 \gtreqless_{\mathcal{L}} s_2$.

($\Leftarrow$) Suppose $s_1 \gtreqless_{\mathcal{L}} s_2$, i.e., $s_1 \ge_{\mathcal{L}} s_2$ or $s_1 \le_{\mathcal{L}} s_2$.

If $s_1 \ge_{\mathcal{L}} s_2$, then $(\exists x \in S^1)$: $xs_1 = s_2$. So for $u = 1$: $xs_1 = us_2$.

If $s_1 \le_{\mathcal{L}} s_2$, then $(\exists u \in S^1)$: $s_1 = us_2$. So for $x = 1$: $xs_1 = us_2$. □

This fact means that S_B satisfies axiom (6L).

We mentioned already that S satisfies also axioms (1) (embedding), (2) (since for relations $(R_1 R_2)^{-1} = R_2^{-1} R_1^{-1}$), (3) and (4) (where here 0 is the empty relation), and (5).

To get axiom (6R) to hold (instead of (6L)) we can use the dual construction of S_B: Let $B^*(S^1)$ be the semigroup of all binary relations on S^1, under composition – but this time we let the relations act on the *left*. It is easy to see that $B^*(S^1)$ is isomorphic to $B(S^1)$, by the isomorphism $R \mapsto R^{-1}$; however $S_{B^*} \le B^*(S^1)$ defined by $S_{B^*} = \langle S \cup \{s^{-1}s \mid s \in S\} \rangle_{B^*(S^1)}$ is not necessarily isomorphic to S_B. This follows from:

2.9. Fact. *Assume S has unambiguous R-order. Then*

$$s_1^{-1}s_2 = 0 \text{ in } S_{B^*} \quad \textit{iff} \quad s_1 \not\lessgtr_{\mathcal{R}} s_2 \text{ in } S.$$

(The proof is dual to that of 2.8.)

Now $S \le S_{B^*}$ and S_{B^*} satisfies axioms (1) through (5) and (6R) – but (6L) does not necessarily hold.

To obtain a semigroup containing S and satisfying all the axioms (1)–(5), (6R and L), we combine S_B and S_{B^*} as follows:

First map the free semigroup $(S \cup \bar{S} \cup \{0\})^+$ onto S_B (and onto S_{B^*}) by: $w \in (S \cup \bar{S} \cup \{0\})^+ \mapsto (w)\varphi \in S_B$, where $(w)\varphi$ is obtained from w by replacing component s by the relation $s \in S \subseteq S_B$, and $\bar{s}$ by $s^{-1} \in S_B$, and 0 by the empty relation; similarly φ^*: $(S \cup \bar{S} \cup \{0\})^+ \twoheadrightarrow S_{B^*}$.

Clearly φ and φ^* are surmorphisms.

From now on *assume that S is unambiguous.*

Next define the relation $\approx$ on $(S \cup \bar{S} \cup \{0\})^+$ by $w_1 \approx w_2$ iff

(1) $(w_1)\varphi = 0$ or $(w_1)\varphi^* = 0$ (i.e., w_1 acts as the empty relation, on the left *or* on the right), *and* $(w_2)\varphi = 0$ or $(w_2)\varphi^* = 0$, or

(2) neither w_1 nor w_2 act as the empty relation (neither left nor right), and $(w_1)\varphi = (w_2)\varphi$ and $(w_1)\varphi^* = (w_2)\varphi^*$.

2.10. Claim. $\approx$ *is a congruence on* $(S \cup \bar{S} \cup \{0\})^+$. *Denote* $(S \cup \bar{S} \cup \{0\})^+/\approx$ *by* $S_\approx$.

Proof. Reflexivitiy and symmetry are obvious. It is also easy to see that $\approx$ is compatible with left and right multiplication in $(S \cup \bar{S} \cup \{0\})^+$. Transitivity is easily showed as follows: let $w_1 \approx w_2$, $w_2 \approx w_3$; from the definition, either w_1, w_2, w_3 never act as 0 (neither left nor right), or each of w_1, w_2, w_3 acts as 0 (on the left or the right – not necessarily all on the same side). If w_1, w_2, w_3 never act as zero, then (by definition of $\approx$) $(w_1)\varphi = (w_2)\varphi$, $(w_2)\varphi = (w_3)\varphi$ and $(w_1)\varphi^* = (w_2)\varphi^*$, $(w_2)\varphi^* = (w_3)\varphi^*$; hence $w_1 \approx w_3$. If w_1, w_2, w_3 can act as 0, then $[(w_1)\varphi = 0$ or $(w_1)\varphi^* = 0]$, and $[(w_2)\varphi = 0$ or $(w_2)\varphi^* = 0]$, and $[(w_3)\varphi = 0$ or $(w_3)\varphi^* = 0]$, hence $w_1 \approx w_3$.

2.11. Claim. $S_\approx$ *satisfies all the axioms* (1)–(5) *and* (6R), (6L). (Recall that we assume that S is unambiguous.)

Proof. (1) $s_1 s_2 \approx (s_1 \cdot s_2)$ since $(s_1 s_2)\varphi = ((s_1 \cdot s_2))\varphi$ and $(s_1 s_2)\varphi^* = ((s_1 \cdot s_2))\varphi^*$.

(2) and (5) are proved similarly.

(4) $s0 \approx 0$ since $(s0)\varphi = 0$ and $(0)\varphi = 0$; the rest of (4), as well as (3), are proved similarly.

(6L) If $s_1 \not\lessgtr_{\mathcal{L}} s_2$, then $(s_1\bar{s}_2)\varphi = s_1 s_2^{-1} = 0$ in S_B (by 2.8), thus $s_1\bar{s}_2 \approx 0$ by the definition of $\approx$.

(6R) If $s_1 \not\lessgtr_{\mathcal{R}} s_2$, then $(\bar{s}_1 s_2)\varphi^* = 0$ in S_{B^*} (by 2.9), hence $\bar{s}_1 s_2 \approx 0$, by the definition of $\approx$.

2.12. Claim. *$S \leq S_{\approx}$ and $\{s^{-1} | s \in S\} \leq S_{\approx}$.*

Proof. If $s_1 \approx s_2$, then (since $s_1, s_2 \in S$ never act as 0 in S_B or S_{B^*}): $(s_1)\varphi = (s_2)\varphi$ and $(s_1)\varphi^* = (s_2)\varphi^*$. Applying these to the element $1 \in S^1$ we obtain $s_1 = 1 \cdot s_1 = (1)((s_1)\varphi) = (1)((s_2)\varphi) = 1 \cdot s_2 = s_2$. So $s_1 = s_2$. If $\bar{s}_1 \approx \bar{s}_2$, then (since they never act as 0): $(\bar{s}_1)\varphi = (\bar{s}_2)\varphi$, thus $s_1^{-1} = s_2^{-1}$; hence (by a remark in 1.2.2), $s_1 = s_2$.

2.13. Fact. *Any semigroup generated by $S \cup \bar{S} \cup \{0\}$ and satisfying all the axioms (1)–(6) is a homomorphic image of $(S)_{\text{reg}}$.*

Proof. Assume T is generated by $S \cup \bar{S} \cup \{0\}$ and satisfies the axioms. The surmorphism $h : (S)_{\text{reg}} \twoheadrightarrow T$ is defined by associating to a product of generators in $(S)_{\text{reg}}$ the same product of generators in T. This is a function: if two words $w_1, w_2 \in (S \cup \bar{S} \cup \{0\})^+$ are the same when considered as products of generators in $(S)_{\text{reg}}$, then w_1, w_2 must also be the same in T, since T satisfies the axioms (1)–(6).

Moreover h is a morphism, by the definition. □

2.14. Corollary. *$S_{\approx}$ is a homomorphic image of $(S)_{\text{reg}}$.*

2.15. Fact. *The homomorphism $h : (S)_{\text{reg}} \twoheadrightarrow S_{\approx}$ is injective when restricted to the subsemigroup generated by S.*

Moreover, this semigroup generated by S in $(S)_{\text{reg}}$, is isomorphic to the semigroup S (i.e., the morphism $s \in S \mapsto s \in (S)_{\text{reg}}$ is one-one).

The dual result holds for $\bar{S}$.

Proof. Let $s_1, s_2 \in \langle S \rangle \subseteq (S)_{\text{reg}}$ and supose $(S_1)h \approx (s_2)h$; then $s_1 \approx s_2$, which implies (by the proof of the claim "$S \leq S_{\approx}$") that $s_1 = s_2$ (same element in the set S). Similarly: $(\bar{s}_1)h \approx (\bar{s}_2)h$ implies $s_1^{-1} = s_2^{-1}$, hence $s_1 = s_2$.

2.16. Corollary. *$S \leq (S)_{\text{reg}}$ and $\bar{S} \leq (S)_{\text{reg}}$.*

Instead of the 'two-sided' relation $\approx$, we can define the 'one-sided' congruences $\approx_{\mathscr{R}}$ and $\approx_{\mathscr{L}}$ on $(S \cup \bar{S} \cup \{0\})^+$, as follows: $w_1 \approx_{\mathscr{R}} w_2$ iff

(1) $[(w_1)\varphi = 0$ or $(w_1)\varphi^* = 0]$ *and* $[(w_2)\varphi = 0$ or $(w_2)\varphi^* = 0]$, **or**

(2) neither w_1 nor w_2 ever act as the empty relation (neither on the left, in S_{B^*}, nor the right, in S_B), and $(w_1)\varphi = (w_2)\varphi$ in S_B.

The relation $\approx_{\mathscr{L}}$ is defined dually.

Remark. $\approx_{\mathscr{R}}$ (resp. $\approx_{\mathscr{L}}$) can also be considered as a relation defined on S_B (resp. S_{B^*}) – instead of $(S \cup \bar{S} \cup \{0\})^+$.

2.17. Claim. *$\approx_{\mathscr{R}}$ is a congruence on $(S \cup \bar{S} \cup \{0\})^+$, and on S_B (and dually for $\approx_{\mathscr{L}}$). Denote $(S \cup \bar{S} \cup \{0\})^+ / \approx_{\mathscr{R}}$ $(= S_B / \approx_{\mathscr{R}})$ by $S_{\approx_{\mathscr{R}}}$ (resp. $S_{\approx_{\mathscr{L}}}$).*

Proof. Same as for the previous claim, for $S_{\approx}$.

2.18. Corollary. *$S_{\approx_\mathcal{R}}$ and $S_{\approx_\mathcal{L}}$ are homomorphic images of $S_{\approx}$ (and hence of $(S)_{\text{reg}}$). Moreover, the homomorphisms are injective when restricted to the subsemigroup S (or the subsemigroup $\{s^{-1} \mid s \in S\}$).*

2.19. Claim. *$S_{\approx_\mathcal{R}}$ and $S_{\approx_\mathcal{L}}$ satisfy all the axioms* (1)–(6). (The proof is the same as for $S_{\approx}$).

2.20. Claim. $S_{\approx} = (S_{\approx_\mathcal{R}} \times S_{\approx_\mathcal{L}})_{S \cup \bar{S} \cup \{0\}}$ (the product in the category of semigroups generated by $S \cup \bar{S} \cup \{0\}$, see [1, 1.6] for definitions).

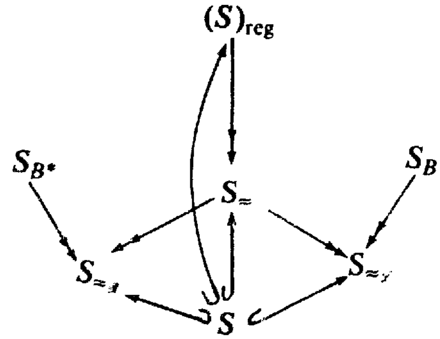
Proof. To $[w]_{\approx} \in S_{\approx}$ associate $([w]_{\approx_\mathcal{R}}, [w]_{\approx_\mathcal{L}}) \in (S_{\approx_\mathcal{R}} \times S_{\approx_\mathcal{L}})_{S \cup \bar{S} \cup \{0\}}$.

If $[w]_{\approx}$ acts as 0 on the left or the right, then $[w]_{\approx} = [w]_{\approx_\mathcal{R}} = [w]_{\approx_\mathcal{L}}$.

If $[w]_{\approx}$ never acts as zero (left or right action), then $[w]_{\approx_\mathcal{R}}$ and $[w]_{\approx_\mathcal{L}}$ together determine $[w]_{\approx}$ (by definition of $\approx$).

Finally we have the following commutative diagram:

2.21



2.22. Fact. *The semigroups $S_{\approx}$, $S_{\approx_\mathcal{R}}$, $S_{\approx_\mathcal{L}}$ are regular. If S is finite, then they are finite and their subgroups divide subgroups of S.*

(Recall the definition of semigroup division: A divides B, denoted $A \prec B$, iff some subsemigroup of B maps homomorphically onto A.)

Proof. Since $S_{\approx}$, $S_{\approx_\mathcal{R}}$, $S_{\approx_\mathcal{L}}$ are homomorphic images of $(S)_{\text{reg}}$, the fact follows from the regularity of $(S)_{\text{reg}}$ – and, in the finite case, from Theorem 2.23 which will be given next. □

Remark. $S_{\approx}$ is not necessarily isomorphic to $(S)_{\text{reg}}$. For example if S is a finite monoid and its (unique) maximal J-class is a non-trivial group G (the 'group of units'), then $s^{-1} \in G \subseteq S$ (for $s \in G$); however in $(S)_{\text{reg}}$, $\bar{s} \notin S$ (as we shall prove later).

2.5. Properties of $(S)_{\text{reg}}$, and of the embedding of $S \leq (S)_{\text{reg}}$

2.23. Theorem. *Let S be an unambiguous semigroup, and let $(S)_{reg}$ be the semigroup constructed in Section 2.2 (and let 0 be the zero of $(S)_{reg}$). Then $S \leq (S)_{reg}$* (Corollary 2.16), *and $(S)_{reg}$ has the following properties:*

(1) *$(S)_{reg}$ is regular* (Fact 2.4).

(2) *The L (resp. R or J) -order or $(S)_{reg}$, restricted to elements of S, is the L (resp. R or J) -order of S. (i.e., if $s_1, s_2 \in S$ and $s_1 \leq s_2$ in $(S)_{reg}$, then $s_1 \leq s_2$ in S, where $\leq$ stands for any one of $\leq_{\mathcal{L}}$, $\leq_{\mathcal{R}}$ or $\leq_{\mathcal{J}}$).*

(3) *Every D-class (resp. J-class) of $(S)_{reg}$, except the J-class $\{0\}$, contains one and only one D-class (resp. J-class) of S.*

Precisely: a D-class of $(S)_{reg}$ is obtained from the D-class of S which it contains, by "adding rows and columns" in the Green–Rees picture.

In particular this implies that an H-class of $(S)_{reg}$ lies either entirely in S (and is an H-class of S), or does not intersect S.

This implies that every group of $(S)_{reg}$ is either a subgroup of S or a Schützenberger group of a non-regular D-class of S – and thus divides S.

And: if every non-regular H-class of S is a singleton, then every subgroup of $(S)_{reg}$ is isomorphic to a subgroup of S.

(4) *If S is finite, then $(S)_{reg}$ is finite. If S is finite-J-above, then $(S)_{reg}$ is finite-J-above except at zero (i.e., $\forall x \neq 0$ in $(S)_{reg}$, the set $\{w \in (S)_{reg} \mid w \geq_{\mathcal{J}} x\}$ is finite).*

If S is infinite, then S and $(S)_{reg}$ have the same cardinaltiy.

(5) *If S is torsion (resp. aperiodic, resp. bounded torsion satisfying $x^{a+b} = x^a$), then $(S)_{reg}$ is torsion (resp. aperiodic, resp. bounded torsion satisfying $x^{1+a+b} = x^{1+a}$).*

Remark. The restriction that S be an unambiguous semigroup is not very strong, since by Theorem 2.1 we have: for any semigroup S there exists a semigroup $\hat{S}_A^+$ such that $\eta : \hat{S}_A^+ \twoheadrightarrow S$; $\hat{S}_A^+$ is unambiguous and its non-regular H-classes are singletons; the morphism η preserves important properties of S, regarding the subgroups and regular elements; if S is finite (resp. finite-J-above), then so is $\hat{S}_A^+$.

Proof of 2.23. (1) We proved already in Corollary 2.16 that $S \leq (S)_{reg}$, and in Fact 2.4 that $(S)_{reg}$ is regular. That $S \leq (S)_{reg}$ also follows from Lemma 2.26.

(3) We shal prove next that every D-class of $(S)_{reg}$, except the J-class $\{0\}$, contains elements of S. This will follow from:

2.24. Fact. *Any non-zero element $x \in (S)_{reg}$ is D-equivalent to the center ($\in S \cup \bar{S}$) of any normal form that represents x.*

Proof. Let $w = s_1 \bar{t}_1 s_2 \bar{t}_2 \cdots \bar{t}_{k-1} s_k \bar{t}_k \cdots s_{n-1} \bar{t}_{n-1} s_n$ with

$$s_1 >_{\mathcal{L}} t_1 >_{\mathcal{R}} s_2 >_{\mathcal{L}} t_2 >_{\mathcal{R}} \cdots >_{\mathcal{L}} t_{k-1} \geq_{\mathcal{R}} s_k \leq_{\mathcal{L}} t_k <_{\mathcal{R}} \cdots <_{\mathcal{R}} s_{n-1} <_{\mathcal{L}} t_{n-1} <_{\mathcal{R}} s_n,$$

be a normal form representing $x \in (S)_{reg}$, $x \neq 0$. (If the center is in $\bar{S}$, the reasoning is almost identical.)

We claim that $x \equiv_{\mathscr{R}} s_1\bar{t}_1 s_2\bar{t}_2 \cdots \bar{t}_{k-1} s_k$. The ordering $\leq_{\mathscr{R}}$ is obvious; the $\geq_{\mathscr{R}}$-ordering is proved by induction on $n-k$ (i.e., the length of the part of w which is right of the center).

If $n-k=0$, then $x = s_1\bar{t}_1 s_2\bar{t}_2 \cdots \bar{t}_{k-1} s_k$.

In general: since $t_{n-1} <_{\mathscr{R}} s_n$ and $s_{n-1} <_{\mathscr{L}} t_{n-1}$, there exists $v, u \in S$ with $t_{n-1} = s_n u$ and $s_{n-1} = v t_{n-1}$. Then

$$\begin{aligned} x \cdot u &= s_1\bar{t}_1 s_2\bar{t}_2 \cdots s_k \cdots s_{n-1}\bar{t}_{n-1} \underbrace{s_n \cdot u} \\ &= s_1\bar{t}_1 s_2\bar{t}_2 \cdots s_k \cdots v \underbrace{t_{n-1} \cdot \bar{t}_{n-1} \cdot t_{n-1}} \\ &= s_1\bar{t}_1 s_2\bar{t}_2 \cdots s_k \cdots v \qquad t_{n-1} \\ &= s_1\bar{t}_1 s_2\bar{t}_2 \cdots s_k \cdots s_{n-1}. \end{aligned}$$

Thus, xu is obtained from x by simply removing $\bar{t}_{n-1} \cdot s_n$; this implies $xu \geq_{\mathscr{R}} x$. Moreover: $x \cdot u \leq_{\mathscr{R}} x$. Hence $xu \equiv_{\mathscr{R}} x$.

Proceeding inductively we obtain $x \equiv_{\mathscr{R}} s_1\bar{t}_1 s_2\bar{t}_2 \cdots \bar{t}_{k-1} s_k$.

In a similar way one proves that $s_1\bar{t}_1 s_2\bar{t}_2 \cdots \bar{t}_{k-1} s_k \equiv_{\mathscr{L}} s_k$. This proves that $x \equiv_{\mathscr{D}} s_k$. □

2.25. Corollary. *Every non-zero D-class (hence every non-zero J-class) of* $(S)_{\text{reg}}$, *contains elements of S.*

Proof. By the above fact, every element $x \in (S)_{\text{reg}}$, with $x \neq 0$, is D-equivalent (hence J-equivalent) to either an element of S or an element of $\bar{S}$ (depending on the center of the normal forms representing x). But, by axiom (5), every element of $\bar{S}$ is D-equivalent to an element of S. □

Proof of 2.23 (contd). To prove part (2) of the theorem and to show that every non-zero D (resp. J) -class of $(S)_{\text{reg}}$ contains *at most one* D (resp. J) -class of S, we use the following lemma:

2.26. Lemma. (1) *The element* 0 *of* $(S)_{\text{reg}}$ *cannot be represented by any other normal form.*

(2) *For all* $s_1, s_2 \in S$: $s_1 \neq \bar{s}_2$ *in* $(S)_{\text{reg}}$.

(3) *Let* $x \in (S)_{\text{reg}}$, *with* $x \neq 0$. *Then all normal forms representing x have the same length (as elements of the set* $(S \cup \bar{S})^+$).

(4) *If* $s \in S \leq (S)_{\text{reg}}$, *then the only normal form representing s is s itself.*

Proof. The lemma follows from the uniqueness of *coded* normal forms – and this is defined and proved in the *Appendix.* □

Remark. By (3) of the lemma, a length function is defined for the elements of $(S)_{\text{reg}}$. Clearly: for $x, y \in (S)_{\text{reg}}$: $\text{length}(xy) \leq \text{length}(x) + \text{length}(y)$.

2.27. Fact. *Let $s_1, s_2 \in S$, and let $\leq$ stand for one of $\leq_{\mathcal{L}}$, $\leq_{\mathcal{R}}$, $\leq_{\mathcal{J}}$; let $\equiv$ stand for one of $\equiv_{\mathcal{L}}$, $\equiv_{\mathcal{R}}$, $\equiv_{\mathcal{D}}$, $\equiv_{\mathcal{J}}$. Then*

$$s_1 \leq s_2 \text{ in } (S)_{\text{reg}} \quad \textit{iff} \quad s_1 \leq s_2 \text{ in } S;$$

$$s_1 \equiv s_2 \text{ in } (S)_{\text{reg}} \quad \textit{iff} \quad s_1 \equiv s_2 \text{ in } S.$$

Proof. We consider the case of $\leq_{\mathcal{L}}$; the other ones are very similar.

If $s_1 \leq_{\mathcal{L}} s_2$ in $(S)_{\text{reg}}$, then either $s_1 = s_2$ (and then $s_1 \leq_{\mathcal{L}} s_2$ in S) or there exists $x \in (S)_{\text{reg}}$ with $s_1 = xs_2$. By the above lemma: $x \neq 0$. So x is of the form $x_1 \bar{y}_1 x_2 \bar{y}_2 \cdots \bar{y}_{n-2} x_{n-1} \bar{y}_{n-1} x_n$ with $x_1 >_{\mathcal{L}} y_1 >_{\mathcal{R}} x_2 >_{\mathcal{L}} y_2 >_{\mathcal{R}} \cdots <_{\mathcal{R}} x_{n-1} <_{\mathcal{L}} y_{n-1} <_{\mathcal{R}} x_n$.

Since $s_1 \neq 0$ we have, by the above lemma: $y_{n-1} \not\geq_{\mathcal{R}} x_n s_2$ (otherwise $x \cdot s_2 = 0$ in $(S)_{\text{reg}}$). If $y_{n-1} <_{\mathcal{R}} x_n \cdot s_2$ or if x_n or $\bar{y}_{n-1}$ is the center of x, then $x_1 \bar{y}_1 x_2 \bar{y}_2 \cdots \bar{y}_{n-2} x_{n-1} \bar{y}_{n-1} x_n s_2$ is a normal form. Otherwise $y_{n-1} \geq_{\mathcal{R}} x_n \cdot s_2$ and $(\exists u \in S^1) y_{n-1} \cdot u = x_n s_2$; moreover $(\exists v \in S)\, x_{n-1} = v y_{n-1}$ since $x_{n-1} <_{\mathcal{L}} y_{n-1}$. So

$$\begin{aligned} xs_2 &= x_1 \bar{y}_1 x_2 \bar{y}_2 \cdots \bar{y}_{n-2} v \underbrace{y_{n-1} \bar{y}_{n-1} y_{n-1}}_{} u \\ &= x_1 \bar{y}_1 x_2 \bar{y}_2 \cdots \bar{y}_{n-2} v \quad\; y_{n-1} \quad\; u \\ &= x_1 \bar{y}_1 x_2 \bar{y}_2 \cdots \bar{y}_{n-2} v x_n s_2. \end{aligned}$$

Again $y_{n-2} \not\geq_{\mathcal{R}} v x_n s_2$ (otherwise $x \cdot s_2 = 0$ in $(S)_{\text{reg}}$). If $y_{n-2} <_{\mathcal{R}} v x_n s_2$ or if x_{n-1} or $\bar{y}_{n-2}$ is the center of x, then $x_1 \bar{y}_1 x_2 \bar{y}_2 \cdots \bar{y}_{n-2} v x_n s_2$ is a normal form. Otherwise we continue, inductively. Finally, there exists $i (< n)$ such that $s_1 = xs_2 = x_1 \bar{y}_1 x_2 \bar{y}_2 \cdots \bar{y}_{n-i} t s_2$, for some $t \in S^1$, and $x_1 \bar{y}_1 x_2 \bar{y}_2 \cdots \bar{y}_{n-i} t s_2$ is a non-zero normal form (i.e., $y_{n-i} <_{\mathcal{R}} ts_2$, or: $y_{n-i} \geq_{\mathcal{R}} ts$ and $\bar{y}_{n-i}$ or ts_2 is the center of xs_2).

However, by Lemma 2.26 the element $s_1 \in S$ is itself its unique normal form representation; hence the normal form $x_1 \bar{y}_1 x_2 \bar{y}_2 \cdots \bar{y}_{n-i} t s_2$ must actually be equal to ts_2. Hence $s_1 = ts_2$ with $t \in S^1$, i.e., $s_1 \leq_{\mathcal{L}} s_2$ in S. □

2.28. Corollary. *Every D (resp. J, R, L) -class of $(S)_{\text{reg}}$ contains at most one D (resp. J, R, L) -class of S.*

This corollary together with Corollary 2.25 implies that every non-zero D (resp. J) -class of $(S)_{\text{reg}}$ contains one and only one D (resp. J) -class of S.

Proof of 2.23 (contd.). To finish the proof of part (3) of the theorem we need the following:

2.29. Fact. *Let $s, t \in S$, and let $x \in (S)_{\text{reg}}$ be such that $s \leq_{\mathcal{R}} x$, $t \leq_{\mathcal{L}} x$ in $(S)_{\text{reg}}$. Then x is an element of S.*

Proof. Let $x_1 \bar{y}_1 x_2 \bar{y}_2 \cdots x_{n-1} \bar{y}_{n-1} x_n$ be a normal form representing x (since $s \leq_{\mathcal{R}} x \ldots$, we cannot have $x = 0$).

We have $s \leq_{\mathcal{R}} x$ and $t \leq_{\mathcal{L}} x$; if $s = x$ or $t = x$, then $x \in S$. In the other case: there exist non-zero elements $u, v \in (S)_{\text{reg}}$ such that $s = xu$ and $t = vx$.

Then the normal form representation of vx is of the form $w\bar{y}_h x_{h+1} \cdots x_{n-1}\bar{y}_{n-1}x_n$, where $w \in (S)_{\text{reg}}$ and $\bar{y}_h x_{h+1} \cdots x_{n-1}\bar{y}_{n-1}x_n$ is the part of x that is right of the center (including the center if center $= \bar{y}_h \in \bar{S}$ and $y_h <_{\mathscr{R}} x_{h+1}$). This follows from an inductive reasoning that is very similar to those used in the proofs of previous properties of normal forms.

However, since $t \in S$, the normal form vx must have length $= 1$ (by Lemma 2.26). Hence the part $\bar{y}_h x_{h+1} \cdots \bar{y}_{n-1}x_n$ does not exist, and x is of the form $x_1\bar{y}_1 \cdots \bar{y}_{h-1}x_h$ with $x_1 >_{\mathscr{L}} y_1 >_{\mathscr{R}} \cdots >_{\mathscr{L}} y_{h-1} \geq_{\mathscr{R}} x_h$. Hence, in particular, the center of the normal form representing x is $x_h \in S$.

By a similar reasoning, this time using $s \leq_{\mathscr{R}} x$, (hence $s = x_1\bar{y}_1 \cdots x_h w'$) one shows that the part of x which is left of the center is empty: hence x is equal to its center $x_h \in S$. So $x \in S$. □

2.30. Corollary. *If $s \in S$, $x \in (S)_{\text{reg}}$ and $s \leq_{\mathscr{H}} x$ in $(S)_{\text{reg}}$, then $x \in S$.*

From this corollary it follows that every H-class of $(S)_{\text{reg}}$ which intersects S lies entirely within S. Moreover this will then be an H-class of S (by Fact 2.27).

From Fact 2.29 it follows that the Green–Rees picture of a non-zero D-class Δ of $(S)_{\text{reg}}$ is obtained by taking the unique D-class δ of S that Δ contains, and adding rows and columns: δ will appear as a full rectangle within Δ – as displayed in Fig. 3.

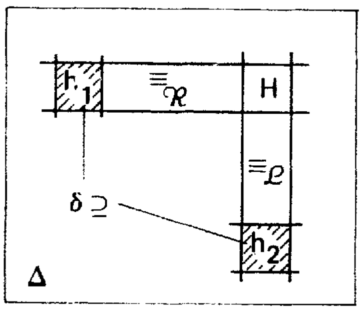


Fig. 3.

If the H-classes $h_1, h_2 \subseteq S$ belong to the D-class $\delta \subseteq S$ and if the H-class H of $\Delta \subseteq (S)_{\text{reg}}$ is $\equiv_{\mathscr{R}} h_1$ and $\equiv_{\mathscr{L}} h_2$ in $(S)_{\text{reg}}$, then $H \subseteq \delta$, by Fact 2.29.

The statements on the groups of $(S)_{\text{reg}}$ follow easily now.

Proof of 2.23 (4). That $(S)_{\text{reg}}$ is *finite* if S is finite follows from the normal form representation.

If S is *infinite*, then $(S)_{\text{reg}}$ has the same cardinality as S, since $S \leq (S)_{\text{reg}}$, and $(S)_{\text{reg}}$ is a homomorphic image of the free semigroup $(S \cup \bar{S} \cup \{0\})^+$, which has the same cardinality as S.

Suppose now that S is *finite-J-above*. To show that for every non-zero element $x \in (S)_{\text{reg}}$ the set $J(\geq x \text{ in } (S)_{\text{reg}}) = \{w \in (S)_{\text{reg}} \mid w \geq_{\mathscr{J}} x \text{ in } (S)_{\text{reg}}\}$ is finite, it is enough to show that $(\forall s \in S)$: the set $J(\geq s \text{ in } (S)_{\text{reg}})$ is finite (since by Corollary 2.25, x is $\equiv_{\mathcal{J}}$ to an element in S).

Let $w \in (S)_{\text{reg}} - \{0\}$ be such that $w \geq_{\mathcal{J}} s$ in $(S)_{\text{reg}}$. So, there exist $w_1, w_2 \in (S)_{\text{reg}}$ such that $w_1 w w_2 = s$. Let $u \in S \cup \bar{S}$ be the center of w (in some representation by a normal form). Then it follows from the definition of normal forms, that u (or $\bar{u}$) is $\geq_{\mathcal{J}}$-above the center of $w_1 w w_2$; moreover the center of $w_1 w w_2$ must be equal to s (by Lemma 2.26).

Thus we proved that if $w \geq_{\mathcal{J}} s$ in $(S)_{\text{reg}}$, where $s \in S$, $w \in (S)_{\text{reg}} - \{0\}$, then the center of w is $s_k \in S$ or $\bar{s}_k \in \bar{S}$ with $s_k \geq_{\mathcal{J}} s$.

Hence, since S is finite-J-above, there are only finitely many possible choices for *centers* of elements $w \in (S)_{\text{reg}}$ such that $w \geq_{\mathcal{J}} s$. Moreover, by the shape of normal forms, there are only finitely many elements in $(S)_{\text{reg}}$ with a given center if S is finite-J-above (recall that the components of a normal form satisfy $\cdots >_{\mathcal{J}} >_{\mathcal{R}} >_{\mathcal{J}} \cdots \geq \text{center} \leq \cdots <_{\mathcal{J}} <_{\mathcal{R}} < \cdots$).

This shows that for every element $s \in S$, the set $\{w \in (S)_{\text{reg}} / w \geq_{\mathcal{J}} s \text{ in } (S)_{\text{reg}}\}$ is finite; hence the same holds true for any element $x \in (S)_{\text{reg}}$, $x \neq 0$ (as mentioned earlier).

Remark. If S is infinite, the element 0 has infinitely many elements $\geq_{\mathcal{J}}$-above (since $\forall s \in S$: $s \geq_{\mathcal{J}} 0$ in $(S)_{\text{reg}}$).

This proves the theorem. □

Remark. By the theorem, unambiguous semigroups are 'close' to regular ones.

Conversely, if S is regular (and A is a set of generators) then $\hat{S}_A^{\mathcal{L}}$ and $\hat{S}_A^{\mathcal{R}}$ are *unambiguous,* and regular (this was observed by J. Rhodes).

(*Proof.* $\hat{S}_A^{\mathcal{R}}$ has unambiguous L-order, $\hat{S}_A^{\mathcal{R}}$ has unambiguous R-order; moreover, the canonical morphism η; $\hat{S}_A^{\mathcal{R}} \twoheadrightarrow \hat{S}_A^{\mathcal{R}}$ is an R^*-morphism, which implies that η preserves the R-structure of regular semigroups. Hence, if S is regular ($\Rightarrow \hat{S}_A^{\mathcal{R}}$ regular), then $\hat{S}_A^{\mathcal{R}}$ has also an unambiguous R-order). See also [1].

2.6. Preservation of (bounded) torsion. Length of products

2.6.1. Torsion and bounded torsion

2.30. Proposition. *If S is torsion (resp. aperiodic), then $(S)_{\text{reg}}$ is torsion (resp. aperiodic).*

More precisely: Suppose every element s of S satisfies one of the identities $s^{a+b} = s^a$, where (a, b) ranges over some subset X of $\mathbb{N} \times \mathbb{N}$; then every element $w \in (S)_{\text{reg}}$ satisfies one of the identities $w^{1+a+b} = w^{1+a}$, where still (a, b) ranges over the set X.

In particular, if S is bounded torsion satisfying the identity $x^{a+b} = x^a$ (for a fixed *pair (a,b)), then $(S)_{reg}$ is also bounded torsion and satisfies the identity $x^{1+a+b}=x^{1+a}$.*

Remark. We are still assuming that S is unambiguous. If S is not unambiguous, then by Theorem 2.1: if S is torsion (every element of S satisfying some torsion identity $x^{a+b}=x^a$, $(a,b)\in X$ for a certain set $X\subseteq \mathbb{N}\times\mathbb{N}$), then $\hat{S}_A^+$ is torsion (with the *same* set of torsion identities).

Proof of 2.30. Assume every element s of S satisfies an identity $s^{a+b}=s^a$, for $(a,b)\in X$ (where X is a given subset of $\mathbb{N}\times\mathbb{N}$, depending only on S).

Let us first prove the proposition for elements $w\in(S)_{reg}$ of length 0, 1, 2 or 3.

The element 0 of $(S)_{reg}$ trivially satisfies any identity of the form $x^{1+a+b}=x^{1+a}$ (no matter what set (a,b) is taken from). Also, if an element $s\in S\leq(S)_{reg}$ satisfies $s^{a+b}=s^a$, then it also satisfies $s^{1+a+b}=s^{1+a}$. This takes care of elements of length 0 or 1.

Assume the *length of w is* 2. Then w is of the form $a\cdot\overline{ba}$, or $ba\cdot\bar{a}$ or $\bar{a}\cdot ab$, or $\overline{ab}\cdot a$ (since we must have L, resp. R-comparability). We shall only consider the case $w=a\cdot\overline{ba}$, since the other ones are dual.

If $a\ngeq_{\mathscr{R}} ba$, then $w^2=0$; thus $w^2=w^3=\cdots=w^n$ (for any n), hence w satisfies any identity of the form $w^{1+h+k}=w^{1+h}$ (for any h,k).

If $a\gtreqless_{\mathscr{R}} ba$, then either $a\geq_{\mathscr{R}} ba$ or $a<_{\mathscr{R}} ba$.

If $a\geq_{\mathscr{R}} ba$, then $(\exists c\in S^1)\, ba=ac$. Hence

$$\begin{aligned} w^2 &= a\,\overline{ba}\,a\,\overline{ba}=a\,\overline{ac}\,a\,\overline{ba}=a\bar{c}\,\bar{a}\,a\,\bar{a}\,\bar{b}\\ &= a\bar{c}\,\bar{a}\,\bar{b}=a\,\overline{bac}=a\overline{b^2}a \qquad \text{since } ac=ba. \end{aligned}$$

More generally: $w^n=a\overline{b^n}a$, as is easy to check. Therefore, if S is torsion (resp. aperiodic), w will be torsion (resp. aperiodic); and if $b\in S$ satisfies the identity $x^{h+k}=x^h$, then w will satisfy the same identity, hence also the identity $x^{1+h+k}=x^{1+h}$.

The case $a<_{\mathscr{R}} ba$ cannot arise if S is a torsion semigroup, as follows from the next lemma.

2.31. Lemma. *If S is a torsion semigroup, then it is impossible to have $a<_{\mathscr{R}} ba$ or $a<_{\mathscr{L}} ab$.* (This lemma expresses the fact that torsion semigroups have the so-called 'stability' property.)

Proof. If $a<_{\mathscr{R}} ba$, then $(\exists c\in S^1)\, a=b\,a\,c$; hence $(\forall n>0)\, a=b^nac^n$. Hence: $ba=b^{n+1}ac^n$, $\forall n\geq 0$. If S is torsion, there exist $h,k\geq 1$ with $c^h=c^{h+k}$. Therefore $ba=b^{h+1}ac^h=b^{h+1}ac^{h+1}c^{k-1}=ac^{k-1}$ (since $b^nac^n=a$, $\forall n$). So $ba=ac^{k-1}$, where $c^{k-1}\in S^1$. This however contradicts the strictness of $a<_{\mathscr{R}} ba$. The case $a<_{\mathscr{L}} ab$ is treated similarly. □

Proof of 2.30 (contd.). We could now directly proceed to the inductive step.

However it is useful to deal with the case of length three to see how the torsion bound increases.

Length = 3: The element w has the form $a\bar{b}c$ of any of its dual forms. Then if $ca \not\geq_{\mathcal{L}} b$ or $ca \not\geq_{\mathcal{R}} b$ we have $w^2 = 0$, so $w^{1+h+k} = w^{1+h}$ for any $h, k > 0$.

If $ca \geq_{\mathcal{L}} b$, $ca \geq_{\mathcal{R}} b$ then $(\exists x, y \in S^1)\ b = cax = yca$. Then

$$w^2 = a\bar{b}c\, a\bar{b}c = a\,\overline{cax}\,ca\,\overline{yca}\,c = a\bar{x}\,\overline{ca}\,ca\,\overline{ca}\,\bar{y}\,c$$

$$= a\bar{x}\,\overline{ca}\,\bar{y}\,c = a\,\overline{ycax}\,c = a\,\overline{cax^2}\,c \qquad \text{since } yca = cax.$$

More generally $w^n = a\,\overline{cax^n}\,c$. Hence if S is torsion (resp. aperiodic), then w is torsion (resp. aperiodic), and if $x \in S$ satisfies the identity $x^{h+k} = x^h$, then w satisfies that same identity, hence also $w^{1+h+k} = w^{1+h}$.

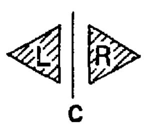
If $ca \leq_{\mathcal{L}} b$, $ca \leq_{\mathcal{R}} b$ then $(\exists x, y \in S^1)\ ca = bx = yb$. Now $w^2 = a\bar{b}ca\bar{b}c$. And:

$$w^3 = a\,\bar{b}\,ca\,\bar{b}\,ca\,\bar{b}\,c = a\bar{b}\,yb\,\bar{b}\,bx\,\bar{b}c$$

$$= a\,\bar{b}\,y\,b\,x\,\bar{b}c = a\bar{b}\,bx^2\,\bar{b}\,c.$$

In general: $w^n = a\bar{b}bx^{n-1}\bar{b}c$. So, if S is torsion (resp. aperiodic), then w is torsion, resp. aperiodic. However, if $x \in S$ satisfies $x^{h+k} = x^h$, then w satisfies $w^{1+h} = w^{1+h+k}$. The other cases ($ca \leq_{\mathcal{L}} b$, $ca >_{\mathcal{R}} b$, etc.) cannot arise if S is torsion (by the above lemma).

Assume w *has length* m, $m \geq 3$. Let w be of the form

w = LcR = 

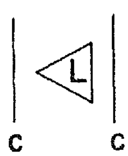
where L is the left side of w, R is the right side of w, and c is the center ($\in S \cup \bar{S}$).

If $w^2 = 0$ then, as already remarked earlier, w will satisfy any equation $w^{1+h+k} = w^{1+h}$, with $h, k > 0$.

So, we will assume from here on that $w^2 \neq 0$. Moreover, for $n \geq 2$ we can write $w^n = L(cRL)^{n-1}cR$. To show that w satisfies an identity $w^{1+h+k} = w^{1+h}$, it is therefore enough to show that $cRLc$ can be written as uc where u is an element of $(S)_{\text{reg}}$ of length ≤ 2. Then indeed $w^n = Lu^{n-1}cR$, and we showed already that every element of length ≤ 2 satisfies a torsion identity of S: $u^{h+k} = u^k$; that way we obtain: $w^{1+h+k} = w^{1+h}$. The proposition will now follow from the following:

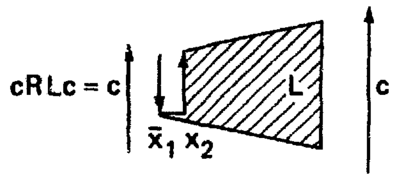
2.32. Claim. *cRLc can be written in the form uc, where u has length at most* 2.

To prove the claim we have to analyze various cases. First we consider the situation where R (or, dually, L) is empty. So:

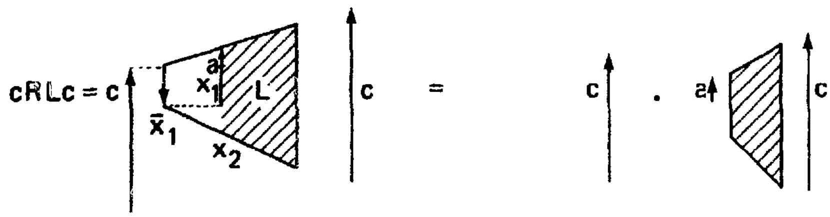
cRLc = 

Four cases arise, according as c belongs to S or $\bar{S}$, and the left-most coordinate of L belongs to S or $\bar{S}$.

Case (A): $c \in S$, and the left-most coordinate of L is $\bar{x}_1 \in \bar{S}$. Then

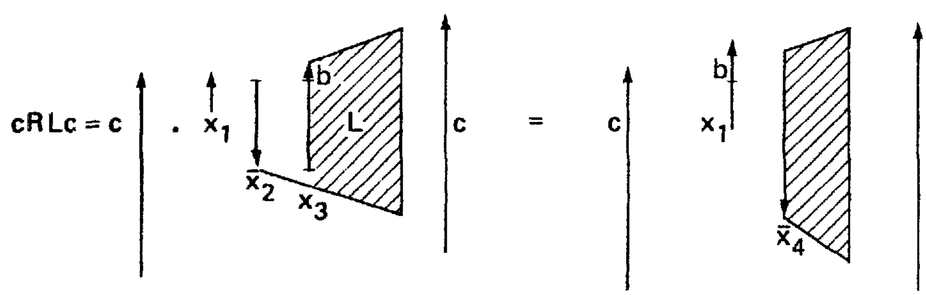


Recall that we assume $w^2 \neq 0$; hence $c \geqq_{\mathcal{J}} x_1$. Also, since Lc is a normal form: $x_1 >_{\mathcal{R}} x_2 >_{\mathcal{J}} \cdots > c$. Hence, since S is torsion (hence stable - or use the lemma proved in this section): $c <_{\mathcal{J}} x_1$. Let $a \in S$ be such that $x_2 = x_1 a$ (since $x_1 >_{\mathcal{R}} x_2$). Then, by reducing:



Thus: Lc has been replaced by a normal form of smaller length.

Case (B): $c \in S$, and the left-most coordinate of L is $x_1 \in S$. Then



(using a similar reasoning as in case (1), since $w^2 \neq 0$ and $x_3 = bx_2 <_{\mathcal{J}} x_2$, etc.; here x_3 could actually be c itself, if $w = LcR = Lc$ has length 3). Again Lc has been replaced by a normal form of smaller length.

Cases (C) *and* (D) (where $c \in \bar{S}$) are dual to cases (A) and (B).

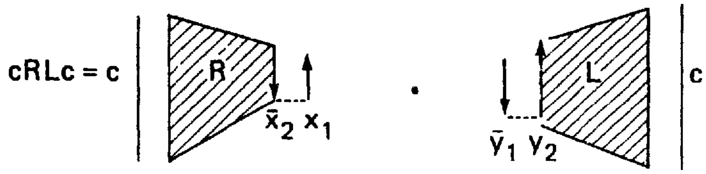
Finally (still in the situation where R is empty), applying cases (A) and (B) (induction on the length of Lc) we obtain: $cRLc = cLc$ has length *one* and can be written as uc (with $u \in S$ if $c \in S$; and $u \in \bar{S}$ if $c \in \bar{S}$).

The case where L is empty is dealt with similarly.

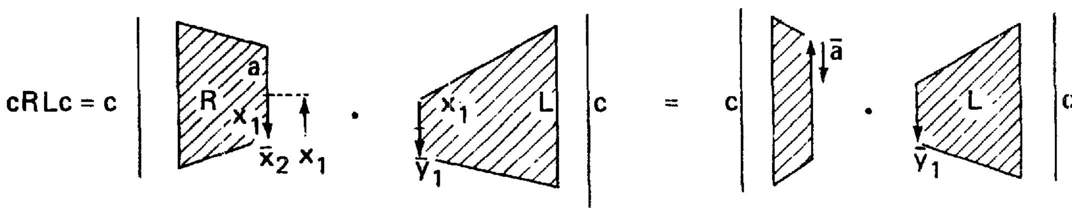
We now consider the situation where neither L nor R are empty, and we shall, by applying reductions, replace Lc or cR by normal forms of smaller length.

Again 4 cases occur according as the left-most coordinate of L and the right-most coordinate of R belong to S or to $\bar{S}$.

Case (1): The right-most coordinate of R is $x_1 \in S$, and the left-most coordinate of L is $\bar{y}_1 \in \bar{S}$. Then

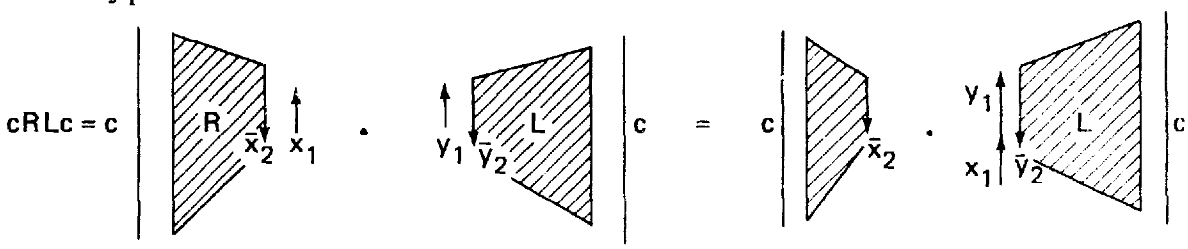


Since $w^2 \neq 0$, we have $x_1 \gtreqless_{\mathscr{Y}} y_1$. Assume $x_1 \geq_{\mathscr{Y}} y_1$ (the other case is dual); let $x_2 = x_1 a$ (x_2 could be c itself). Then



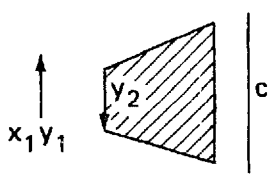
So cR has been replaced by a normal form of smaller length.

Case (2): The right-most coordinate of R is $x_1 \in S$, and the left-most coordinate of L is $y_1 \in S$. Then

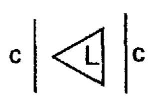


So, cR has been replaced by a normal form of smaller length, while Lc has been replaced by $x_1 Lc$ which is a word that has the same length Lc, but which still has to be reduced.

If $x_1 y_1 >_{\mathscr{Y}} y_2$, then $x_1 Lc$ is already reduced. Assume $x_1 y_1 \leq_{\mathscr{Y}} y_1$; then by applying to

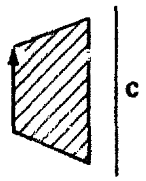


the same reasoning as for the situation



(cases (A), (B), where R is empty), we make the string shorter and shorter. In the end two cases can occur:

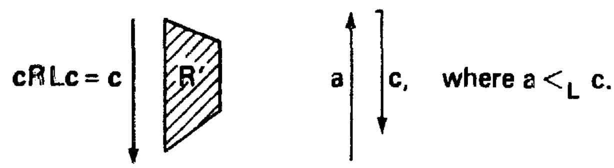
Case (2a): $x_1 Lc$ is finally reduced to a normal form



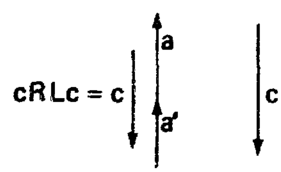
Case (2b): $x_1 Lc$ reduces to a normal form

(under the conditions of case (2) this can only happen if $c \in \bar{S}$); here $a <_L c$. (i.e. c will no longer be the center of the new normal form).

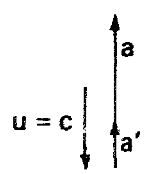
If case (2b) ever occurs we shall have:



Then, applying again the reasoning of cases (A) and (B) (where this time the equivalent of "L" is empty), R' will be replaced by shorter and shorter strings, and finally



So we can write $cRLc = uc$, taking



(length 2).

If case (2b) never occurs, we alternately apply cases (1) and (2a) and make L and R shorter ... until one of L or R disappears (length 0). Then we are back in case (A) or (B). So here $cRLc$ has length one (see cases (A) and (B)), and $cRLc = uc$ (u of length 1).

Further cases occur, when the right-most coordinate of R is $\bar{x}_1 \in \bar{S}$, but these cases are dual to cases (1) and (2).

This proves the claim, and hence the proposition. □

2.6.2. Bound on the length of products

The reasoning that proves that bounded torsion etc. is preserved (when we go from S to $(S)_{\text{reg}}$) can be generalized. If w_1, w_2 are two elements of $(S)_{\text{reg}}$, we can give a good upper bound of the length of $w_1 w_2$ (when represented by a normal form). This product formula connects the two "dimensions" that normal forms have: the *length* (number of coordinates, alternatingly in S and $\bar{S}$), and the '*depth*' (still to be defined) of the center coordinate (see Fig. 4).

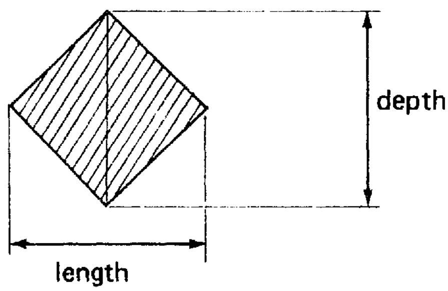


Fig. 4.

In the case of stable semigroups, the J-order plays the role of the depth-order. Recall that a semigroup is 'stable' iff the following holds: let $a \le_{\mathcal{L}} b$ (resp. $\le_{\mathcal{R}}$); then $a <_{\mathcal{L}} b$ (resp. $<_{\mathcal{R}}$) iff $a <_{\mathcal{J}} b$.

Let $l(\cdot)$ denote the length function; let $w_1 = L_1 c_1 R_1$ and $w_2 = L_2 c_2 R_2$ be elements of $(S)_{\text{reg}}$ (where L, R, c denote respectively the left side, the right side, the center).

2.33. Proposition. *Assume S is a stable semigroup. Then:*

(1) *If $c_1 \not>_{\mathcal{J}} c_2$, then $l(w_1 w_2) \le l(w_1) + l(c_2 R_2)$.*

(2) *If $c_1 \not>_{\mathcal{J}} c_2$, and if $c_1 R_1 = c_1 R_1' R_1''$, where R_1' is the set of those coordinates x of R_1 for which $x \not>_{\mathcal{J}} c_2$, then:*

$$l(w_1 w_2) = l(L_1 c_1 R_1') + l(c_2 R_2).$$

Proof. Part (2) of the proposition is obtained by iterating part (1). The proof of part (1) is very similar (case analysis) to the proof of preservation of bounded torsion. □

If S is *not stable*, then the depth condition "$c_1 \not>_{\mathcal{J}} c_2$" is to be replaced with the following:

$$(\forall \alpha, c_1 \ge_{\mathcal{J}} \alpha)\,(\forall x, x \ge_{\mathcal{J}} c_2): \quad \alpha \not>_{\mathcal{R}} x \text{ and } \alpha \not>_{\mathcal{L}} x.$$

It can be checked that if S is stable, these two conditions are equivalent.

For arbitrary semigroups one could define: b is *deeper* than a iff

$$(\exists \alpha, \beta): \quad b \le_{\mathcal{J}} \beta \boxed{<_{\mathcal{R}} \text{ or } <_{\mathcal{L}}} \alpha \le_{\mathcal{J}} \alpha.$$

(If S is stable this is equivalent to $b <_{\mathcal{J}} a$.) Weaker definitions of depth could be devised, and the proposition would still hold.

2.7. *Further properties, and variations of the construction* $(S)_{\text{reg}}$

2.7.1. *Unambiguity except at zero*

Recall the definition of unambiguity (Section 2.1). More generally, we define that an element s is L-unambiguous iff $\forall x, y$: $[x \geq_{\mathcal{L}} s,\ y \geq_{\mathcal{L}} s] \Rightarrow x \gtreqless_{\mathcal{L}} y$. Similarly we define '$R$-unambiguous' and 'unambiguous' (both L and R-unambiguous). 'Ambiguous' means 'not unambiguous'.

One can show easily that the set of L (resp. R) -ambiguous elements of a semigroup forms a left (resp. right) ideal.

If a semigroup contains a zero, then this zero is always both L and R-ambiguous.

Definition. A semigroup is *unambiguous except at zero* iff it is unambiguous, or if it contains a zero and all non-zero elements are unambiguous.

We mentioned earlier (in the first remark of Section 2.2) that if an element $s \in S$ is L or R-ambiguous, then s is identified with 0 in $(S)_{\text{reg}}$. It will follow from the Appendix of this paper that unambiguous elements of S are kept distinct in $(S)_{\text{reg}}$. It follows that in order to have $S \leq (S)_{\text{reg}}$ (with no elements of S identified in $(S)_{\text{reg}}$), it is enough to assume that S is unambiguous, except at zero (here we assume that if S has a zero, this element will also be used as the zero of $(S)_{\text{reg}}$).

In fact more is true:

All the properties of $(S)_{\text{reg}}$ proved so far (and those that will be proved in the Appendix) hold if we only assume that S *is unambiguous except at zero.*

In that case, if S has a zero, no *new* zero has to be added to $(S)_{\text{reg}}$ (but the one of S can be used).

If S has a zero 0 but is not unambiguous, except-at-zero, then $\hat{S}_A^+/(0)\eta^{-1}$ is unambiguous except at zero. (Notation: $\eta: \hat{S}_A^+ \twoheadrightarrow S$ is the canonical morphism (see Section 2.1); $(0)\eta^{-1}$ is an ideal of $\hat{S}_A^+$, and $\hat{S}_A^+/(0)\eta^{-1}$ is the Rees quotient over that ideal).

2.7.2. *Green relations of* $(S)_{\text{reg}}$

The J-order, and the D-relation

2.34. Fact. *Let* w_1 *and* w_2 *be elements of* $(S)_{\text{reg}}$, *represented by normal forms. Let* w_1, w_2 *have respective centers* c_1, c_2; *if* c_1 *(or* c_2*) belongs to* $\bar{S}$, *we write* $c_1 = \bar{s}_1$ *(or* $c_2 = \bar{s}_2$*); if* c_1 *(or* c_2*) belongs to S, we take* $c_1 = s_1$ *(or* $c_2 = s_2$*). Then:*

$$w_1 \leq_{\mathcal{J}} w_2 \text{ in } (S)_{\text{reg}} \quad \textit{iff} \quad s_1 \leq_{\mathcal{J}} s_2 \text{ in } S.$$

$$w_1 \equiv_{\mathcal{J}} w_2 \text{ in } (S)_{\text{reg}} \quad \textit{iff} \quad s_1 \equiv_{\mathcal{J}} s_2 \text{ in } S.$$

So, the $\leq_{\mathcal{J}}$-order and the D-relation are determined by the center of any normal form representation of the elements of $(S)_{\text{reg}}$ (whereby one can even ignore whether the center belongs to S or $\bar{S}$).

Proof. From Fact 2.24 we know that any element of $(S)_{\text{reg}}$ is D-equivalent to its center (in any normal form representation); also $\bar{s} \equiv_{\mathcal{J}} s$. Moreover, by Corollary 2.28, two elements of $(S)_{\text{reg}}$ are *not* D-equivalent if their centers are not D-equivalent (for the D-order of S). The fact then follows. □

The L, R, and H orders

The expression of the L, R, and H orders of $(S)_{\text{reg}}$ in terms of normal forms can only be given by using the *coding* of normal forms. We know (see Fact 2.5) that formally different normal forms may represent the same element of $(S)_{\text{reg}}$. The coding transforms such normal forms into each other, and conversely, if two normal forms represent the same element of $(S)_{\text{reg}}$ they can be coded into each other. This coding is described in the Appendix (see the proof of Fact A.1.2), and it is also shown how unique representatives for all those normal forms representing the same element of $(S)_{\text{reg}}$ can be found. The unique representatives (called 'coded normal forms') are described in (A.1.1); their uniqueness was used in obtaining Lemma 2.26, and the lengthy proof of uniqueness occupies part A2 of the Appendix.

Recall that $(S)_{\text{reg}}$ is a homomorphic image of the free product of S and its reverse $\bar{S}$, with a zero added.

Definition. Let x_1, x_2 be elements of the free product $A \circledast B$ of two semigroups A and B. Then x_1 is a *right subsegment* of x_2 iff $x_1 = x_2$ or $(\exists y \in A \circledast B)\, x_2 = yx_1$. (For example: if $x_2 = a_1 b_1 a_2 b_2 a_3$, then $x_1 = ab_2 a_3$ is a right subsegment of x_2 if $a_2 = a$ or if $a_2 = a'a$ for some $a' \in A$.)

Similarly, one defines left subsegments.

2.35. Fact. *Let $w_1 = L_1 c_1 R_1$ and $w_2 = L_2 c_2 R_2$ be elements of $(S)_{\text{reg}}$, represented by coded normal forms (L, R, c respectively denote the left side, the right side, the center). Then $w_1 \geq_{\mathcal{L}} w_2$ in $(S)_{\text{reg}}$ iff $c_1 R_1$ is a right subsegment of $c_2 R_2$ (when $c_1 R_1, c_2 R_2$ are considered as words belonging to the free product of S and $\bar{S}$).*

Equivalently: Let $w_1 = L_1 c_1 R_1$, $w_2 = L_2 c_2 R_2$ be representations by normal forms (not necessarily coded). Then $w_1 \geq_{\mathcal{L}} w_2$ in $(S)_{\text{reg}}$ iff $c_1 R_1$ can be transformed by the coding producedure into $c_1' R_1'$ which is a right subsegment of $c_2 R_2$ (as words in the free product of S and $\bar{S}$).

The dual statement describes the R-order of $(S)_{\text{reg}}$ in terms of left subsegments.

The H-order is obtained by combining the L and the R-order.

In particular we have (for coded normal forms):

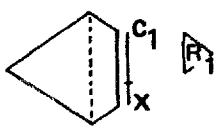
$$w_1 \equiv_{\mathcal{L}} w_2 \quad \textit{iff} \quad R_1 = R_2 \textit{ and } c_1 \equiv_{\mathcal{L}} c_2 \textit{ (in } S \textit{ or } \bar{S}),$$

$$w_2 \equiv_{\mathscr{R}} w_2 \quad \textit{iff} \quad L_1 = L_2 \textit{ and } c_1 \equiv_{\mathscr{R}} c_2 \textit{ (in } S \textit{ or } \bar{S}\textit{)},$$

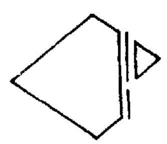
$$w_1 \equiv_{\mathscr{H}} w_2 \quad \textit{iff} \quad L_1 = L_2,\ R_1 = R_2 \text{ and } c_1 \equiv_{\mathscr{H}} c_2 \textit{ (in } S \textit{ or } \bar{S}\textit{)}.$$

Proof. We only consider $\geq_{\mathscr{L}}$ (the other cases are similar).

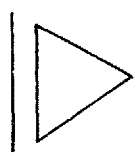
The second formulation (not using coded normal forms) is easily seen to be equivalent to the formulation using coded normal forms. We know (see the proof of Fact 2.24) that $w_1 \equiv_{\mathscr{L}} c_1R_1$ and $w_2 \equiv_{\mathscr{L}} c_2R_2$ in $(S)_{\text{reg}}$. Also, if w_1, w_2 are coded normal forms, then c_1R_1 and c_2R_2 will be coded normal forms. Now, $c_1R_1 \geq_{\mathscr{L}} c_2R_2$ iff $c_1R_1 = c_2R_2$ or $(\exists u \in (S)_{\text{reg}})\ uc_1R_1 = c_2R_2$. After reducing, uc_1R_1 will be represented by a normal form



the part R_1 of the normal form is in coded form already. If we completely code this normal form representing uc_1R_1, we will replace xc_1 by a representative of an L or R-class of S: if $c_1 \in S$, then xc_1 $(\in S)$ will be replaced by an L-equivalent zxc_2 (see the coding procedure in A.1.2); if $c_1 = \bar{s}_1 \in \bar{S}$ and $x = \bar{y} \in \bar{S}$, then s_1y will be replaced by an R-equivalent element s_1yz (so now xc_1 is replaced by $\bar{z}xc_1$). It could also happen that c_1 belongs to the center of uc_1R_1; in that case no coding of c_1 is necessary. After the coding is done, uc_1R_1 will still look like

$uc_1R_1 =$ 

Since this is equal to

$c_2R_2 =$ 

we conclude (by uniqueness of coded normal forms) that c_1R_1 is a right subsegment of c_2R_2.

The converse (if c_1R_1 is a right subsegment of c_2R_2, then $c_1R_1 \geq_{\mathscr{L}} c_2R_2$) is immediate from the definition of the L-order. □

An important consequence is:

2.36. Proposition. *If S is unambiguous except at zero (which we assumed all along), then $(S)_{\text{reg}}$ is unambiguous except at zero.*

Proof. This follows easily from the expression of the L and R order of $(S)_{\text{reg}}$ just given. □

2.7.3. *A variation of the construction $(S)_{\text{reg}}$: New inverses for non-regular elements only*

In the construction $(S)_{\text{reg}}$ we introduced the element $\bar s$, which will be a regular inverse of s, no matter whether s is already regular (and has already inverses in S) or not.

We now give a variation of the construction, and this time we introduce $\bar s$ only if s is non-regular (*Remark.* Even in this case we might still *indirectly* introduce new inverses for regular elements s, namely inverses of the form $s_1\bar n$ etc.)

Let S be a semigroup that is unambiguous (or unambiguous except at zero). Let N be the set of non-regular elements of S, and let $\bar N = \{\bar n \mid n \in N\}$ be a set that is disjoint from S. Let 0 be an additional element that belongs neither to S nor $\bar N$.

We define $(S)_{\text{reg}, N}$ to be the semigroup presented by the generators $S \cup \bar N \cup \{0\}$ and the following relations:

(1) $s_1 s_2 = s_3$ if $s_1 \cdot s_2 = s_3$ in S
(where $\cdot$ denotes the multiplication of S).

(2) $\bar n_1 \bar n_2 = 0$ if $n_1, n_2 \in N$.

(3) 0 is a zero (i.e. $0s = s0 = 0\bar n = \bar n 0 = 0$).

(4) $\bar n_1 s \bar n_2 = 0$ if $n_1 <_{\mathcal{R}} s >_{\mathcal{L}} n_2$, and $n_1, n_2 \in N$, $s \in S$.

(5)(A) $n\bar n n = n$ if $n \in N$,

(B) for every $n_1, n_2 \in N$, $s \in S$ with $n_1 \equiv_{\mathcal{R}} s \geq_{\mathcal{L}} n_2$ or $n_1 \leq_{\mathcal{R}} s \equiv_{\mathcal{L}} n_2$:
$\bar n_1 s \bar n_2 = \overline{usv}$, where $u, v \in S$ are such that $n_1 = sv$, $n_2 = us$.

(6)(L) $s\bar n = 0$ if $s \not\geq_{\mathcal{L}} n$,

(R) $\bar n s = 0$ if $s \not\geq_{\mathcal{R}} n$ $(s \in S, n \in N)$.

Comments on the relations. The relations for $(S)_{\text{reg}, N}$ are similar to those for $(S)_{\text{reg}}$. The differences come from the fact that we want to avoid elements $\bar s$ where s is regular in S. This directly explains relation (2): if n_1, n_2 are non-regular, it could happen that $n_2 n_1$ is regular; hence we must not set $\bar n_1 \bar n_2 = \overline{n_2 n_1}$. Even if $n_2 n_1 \in N$ we define $\bar n_1 \bar n_2 = 0$ (otherwise the following could happen: suppose $n_1, n_2, n_3, n_2 n_1, n_3 n_2 n_1 \in N$, but $n_3 n_2 \notin N$; then $\bar n_2 \bar n_3 = 0$, so $\bar n_1 \bar n_2 \bar n_3 = 0$; but also $\bar n_1 \bar n_2 \bar n_3 = \overline{n_2 n_1}\,\bar n_3 = \overline{n_3 n_2 n_1}$). Relation (4) can be explained similarly.

Relation (5B) (together with the other relations) will enable us to represent elements of $(S)_{\text{reg}, N}$ by normal forms, just like for $(S)_{\text{reg}}$. Intuitively we would want

$$\bar n_1 s \bar n_2 = \overline{sv}\, s\, \overline{us} = \bar v \bar s s \bar s \bar u = \bar v \bar s \bar u = \overline{usv};$$

this computation is not allowed in $(S)_{\text{reg}, N}$, so (5B) simply postulates the result. Notice also that $usv \in N$ under the given conditions: if $n_1 \equiv_{\mathcal{R}} s$, then $un_1 \equiv_{\mathcal{R}} us$ $(= n_2)$; but $un_1 = usv$, so $usv \equiv_{\mathcal{R}} n_2$ $(\in N)$. Similarly, if $n_2 \equiv_{\mathcal{L}} s$, then $usv \equiv_{\mathcal{L}} n_1$ and, of course, nonregularity is preserved under $\equiv_{\mathcal{L}}$ and $\equiv_{\mathcal{R}}$.

As for $(S)_{\text{reg}}$, we can code normal forms of $(S)_{\text{reg}, N}$ to representatives (in L and R-classes; see the Appendix). Indeed the following property of $(S)_{\text{reg}}$ also holds for $(S)_{\text{reg}, N}$.

2.37. Fact. *If $n \in N$ and $an \equiv_{\mathscr{L}} n$, then $\overline{ant} \cdot an = \overline{nt} \cdot n$ (for any $t \in S$ such that $nt \in N$). Dually: if $n\beta \equiv_{\mathscr{R}} n$, then $n\beta \cdot \overline{tn\beta} = n \cdot \overline{tn}$.*

Proof.

$$\overline{ant} \cdot an = \overline{ant} \cdot an\bar{n}n = \overline{ant} \cdot an \cdot \overline{xan}n \qquad \text{where } xan = n,\ n \equiv_{\mathscr{L}} an$$

$$= \overline{xant} \cdot n \qquad \text{by (5B), since } ant \leq_{\mathscr{R}} an \equiv_{\mathscr{L}} xan.$$

The proof of the second statement is similar (dual). □

As in the case of $(S)_{\text{reg}}$ (see Appendix) one can show that coded normal forms are unique in $(S)_{\text{reg},N}$.

In fact all the properties of $(S)_{\text{reg}}$ that are summarized in Theorem 2.23 hold for $(S)_{\text{reg},N}$.

All the proofs are similar. Concerning the regularity of $(S)_{\text{reg},N}$: if $s_1\bar{n}_1s_2\bar{n}_2 \cdots s_k\bar{n}_ks_{k+1}$ is a normal form representing an element of $(S)_{\text{reg},N}$ one can check that the following word represents a regular inverse for that element: $\hat{s}_{k+1}n_k\hat{s}_k \cdots n_2\hat{s}_2n_1\hat{s}_1$, where $\hat{s}_i = \bar{s}_i$ if $s_i \in N$ and $\hat{s}_i$ = any regular inverse of s_i in S, if s_i is regular in S.

Appendix

Unique representation of elements of $(S)_{\text{reg}}$ by coded normal forms

A1. *Coded normal forms*

We saw (in Fact 2.2) that every element of $(S)_{\text{reg}}$ can be represented by 0 or by a word of $(S \cup \bar{S})^+$ in normal form. We also saw (in Fact 2.5) that this representative is not necessarily unique; in this section we shall use Fact 2.5 to code the normal forms, and in Section A2 we shall show that these coded normal forms are unique.

Recall Fact 2.5, where we proved:
if $as \equiv_{\mathscr{L}} s$, then $\overline{as} \cdot as = \bar{s}s$,
if $sb \equiv_{\mathscr{R}} s$, then $sb \cdot \overline{sb} = s\bar{s}$.

Let l (resp. r) be a representative of the L-class (resp. R-class) of s; then by Fact 2.5: $\bar{s}s = \bar{l} \cdot l$ and $s\bar{s} = r \cdot \bar{r}$. This leads to the following definition:

A.1.1. Definition (*coded normal form*). Assume that for every L (resp. R) -class of S a representative has been chosen. This set of representatives is kept fixed in the sequel.

A *coded normal form* is either 0 or a normal form in $(S \cup \bar{S})^+$ of one of the following two kinds (a) or (b):

(a) (center in S): $(r_1)\,\bar{l}_1r_2\bar{l}_2 \cdots \bar{l}_{k-1}c_k\bar{r}_k \cdots l_{n-1}\bar{r}_{n-1}\,(l_n)$

with $(r_1 >_{\mathscr{L}})\, l_1 >_{\mathscr{R}} r_2 >_{\mathscr{L}} l_2 >_{\mathscr{R}} \cdots >_{\mathscr{L}} l_{k-1} \geq_{\mathscr{R}} c_k \leq_{\mathscr{L}} r_k <_{\mathscr{R}} \cdots <_{\mathscr{R}} l_{n-1} <_{\mathscr{L}} r_{n-1}\, (<_{\mathscr{R}} l_n)$

(Again, as in Fact 2.2, the fact that r_1 and l_n are in parentheses indicates that these elements may or may not be present.)

(b) (center in $\bar{S}$): $(r_1)\bar{l}_1 r_2 \bar{l}_2 \cdots r_k \bar{c}_k l_{k+1} \cdots l_{n-1}\bar{r}_{n-1}(l_n)$

$$\text{with } (r_1 <_{\mathscr{L}})\, l_1 >_{\mathscr{R}} r_2 >_{\mathscr{L}} l_2 >_{\mathscr{R}} \cdots >_{\mathscr{R}} r_k >_{\mathscr{L}} c_k <_{\mathscr{R}} l_{k+1} <_{\mathscr{L}} \cdots <_{\mathscr{R}} l_{n-1} <_{\mathscr{L}} r_{n-1} (<_{\mathscr{R}} l_n)$$

with the supplementary condition that $r_1, r_2, \ldots, r_{n-1}$ are among the *fixed representatives of the R-classes* of S, and $l_1, l_2, \ldots, l_n$ are among the *fixed representatives of the L-classes* of S. No new condition is put on c_k $(\in S)$.

A.1.2. Fact. *Assume fixed representatives of the L and R-classes of S have been chosen. Every element of* $(S)_{\text{reg}}$ *can be represented by a coded normal form.*

Proof. We show that every normal form is equivalent to a coded normal form — by induction on the lengths of the sides (left, resp. right of the center) of the given normal forms. The two sides are dealt with independently.

First, normal forms of length 0 or 1 are already coded (since their sides are empty).

Coding of the *right side*: suppose the following normal form is given:

$$x = s_1\bar{t}_1 \cdots \bar{t}_{k-1} s_k \bar{t}_k \cdots s_{n-1}\bar{t}_{n-1} s_n;$$

let $s_n \equiv_{\mathscr{L}} l_n$ (representative of L-class); so $s_n = ul_n$ $(u \in S^1)$; also $t_{n-1} <_{\mathscr{R}} s_n$, so $t_{n-1} = s_n b_n = ul_n b_n$ (for some $b_n \in S^1$). Hence

$$\begin{aligned} x &= s_1\bar{t}_1 \cdots \bar{t}_{k-1} s_k \bar{t}_k \cdots s_{n-1} \cdot \overline{ul_n b_n} \cdot ul_n \\ &= s_1\bar{t}_1 \cdots \bar{t}_{k-1} s_k \bar{t}_k \cdots s_{n-1} \bar{b}_n \cdot \overline{ul_n} ul_n \\ &= s_1\bar{t}_1 \cdots \bar{t}_{k-1} s_k \bar{t}_k \cdots s_{n-1} \bar{b}_n \bar{l}_n l_n \qquad \text{(by Fact 2.5(a))} \\ &= s_1\bar{t}_1 \cdots \bar{t}_{k-1} s_k \bar{t}_k \cdots s_{n-1} \overline{l_n b_n} l_n. \end{aligned}$$

Thus, the right-most component of the right side has been coded.

Remark 1. The result does not depend on the choice of u and b_n such that $s_n = ul_n$ and $t_{n-1} = s_n b_n$. Indeed, u is eliminated in the result, and if $t_{n-1} = s_n b_n = s_n b_n'$ then (letting $l_n = ds_n \equiv_{\mathscr{L}} s_n$) we have $l_n b_n = ds_n b_n = ds_n b_n' = l_n b_n'$.

Remark 2. The result is again a normal form, i.e., $s_{n-1} <_{\mathscr{L}} l_n b_n <_{\mathscr{R}} l_n$. Indeed $t_{n-1} \equiv_{\mathscr{L}} l_n b_n$ (since $s_n \equiv_{\mathscr{L}} l_n \Rightarrow s_n b_n \equiv_{\mathscr{L}} l_n b_n$, and $t_{n-1} = s_n b_n$), so $s_{n-1} <_{\mathscr{L}} l_n b_n$ $(\equiv_{\mathscr{L}} t_{n-1})$.

Also $l_n b_n \leq_{\mathscr{R}} l_n$, and $l_n b_n \not\equiv_{\mathscr{R}} l_n$, otherwise $(\exists \alpha)\, l_n b_n \alpha = l_n$, which would imply $ul_n b_n \alpha = ul_n$, hence (since $s_n = ul_n$): $s_n b_n \alpha = s_n$, hence (since $s_n b_n = t_{n-1}$): $t_{n-1}\alpha = s_n$; this however contradicts $t_{n-1} <_{\mathscr{R}} s_n$. (End of Remark 2.)

The same procedure can be continued:

$$\begin{aligned} x &= s_1\bar{t}_1 \cdots \bar{t}_{k-1} s_k \bar{t}_k \cdots \bar{t}_{n-2} s_{n-1} \overline{l_n b_n} l_n \\ &= s_1\bar{t}_1 \cdots \bar{t}_{k-1} s_k \bar{t}_k \cdots \bar{t}_{n-2} \cdot a_n l_n b_n \overline{l_n b_n} l_n, \end{aligned}$$

where $s_{n-1}=a_n l_n b_n$ for $a_n \in S$ (since $s_{n-1} <_{\mathscr{J}} l_n b_n$);

$$=s_1\bar{t}_1 \cdots \bar{t}_{k-1} s_k \bar{t}_k \cdots \bar{t}_{n-2} \cdot a_n r_{n-1} v \cdot \overline{r_{n-1} v} \cdot l_n,$$

where r_{n-1} is the representative of the R-class of $l_n b_n$: $r_{n-1} \equiv_{\mathscr{R}} l_n b_n$, $r_{n-1} v = l_n b_n$ for $v \in S^1$; etc.

$$=s_1\bar{t}_1 \cdots \bar{t}_{k-1} s_k \bar{t}_k \cdots \bar{t}_{n-2} \cdot a_n r_{n-1} \bar{r}_{n-1} l_n, \qquad \text{(by Fact 2.5(b))}.$$

We can again observe (as in the above Remarks 1 and 2) that the result does not depend on the choice of v and a_n (but only on s_{n-1} and $l_n \cdot b_n$), since v is eliminated, and if $s_{n-1}=a_n l_n b_n = a'_n l_n b_n$ then, (letting $r_{n-1} = l_n b_n h \equiv_{\mathscr{R}} l_n b_n$) we have $a_n r_{n-1} = a_n l_n b_n h = a'_n l_n b_n h = a'_n r_{n-1}$.

And, the result is a normal form, i.e., $t_{n-2} <_{\mathscr{R}} a_n r_{n-1} <_{\mathscr{J}} r_{n-1} <_{\mathscr{R}} l_n$. Indeed $s_{n-1} \equiv_{\mathscr{R}} a_n r_{n-1}$ (since $l_n b_n \equiv_{\mathscr{R}} r_{n-1} \Rightarrow a_n l_n b_n \equiv_{\mathscr{R}} a_n r_{n-1}$, and $s_{n-1} = a_n l_n b_n$); hence $t_{n-2} <_{\mathscr{R}} a_n r_{n-1}$ ($\equiv_{\mathscr{R}} s_{n-1}$). Also $a_n r_{n-1} \leq_{\mathscr{J}} r_{n-1}$, and $a_n r_{n-1} \not\equiv_{\mathscr{J}} r_{n-1}$; otherwise $(\exists y)\, y a_n r_{n-1} = r_{n-1}$, which would imply $y a_n r_{n-1} v = r_{n-1} v$ hence (since $r_{n-1} v = l_n b_n$): $y a_n l_n b_n = l_n b_n$; thus (since $a_n l_n b_n = s_{n-1}$): $y s_{n-1} = l_n b_n$ – which contradicts $s_{n-1} <_{\mathscr{J}} l_n b_n$ established in Remark 2. Finally $r_{n-1} <_{\mathscr{R}} l_n$ since $r_{n-1} \equiv_{\mathscr{R}} l_n b_n$ and $l_n b_n <_{\mathscr{R}} l_n$ (established in Remark 2).

Continuing inductively, we code the whole right side of the normal form. In the same way, the left side is coded.

Also, the result does not depend on which side was coded first.

If the center of the normal form is in $\bar{S}$ the same proof applies.

This proves Fact A1.2.

Remark. The above coding of normal forms is probably related to the '*Zeiger coding*' (see [4]).

A2. *Uniqueness of coded normal forms*

Proposition. *Every element of* $(S)_{\text{reg}}$ *can be represented by one and only one coded normal form (for a given choice of representatives of the L and R-classes of S).* (*Remark.* We still assume S is unambiguous.)

That every element of $(S)_{\text{reg}}$ can be represented by at least one coded normal form was proved in A.1 – always keeping a fixed set of representatives of the L and R-classes of S.

To show *uniqueness* we let $(S)_{\text{reg}}$ act on a certain set of *states* (transformation semigroup) and show that elements which are represented by different coded normal forms act differently on those states. More precisely:

Choose as state set Q the set of all coded normal forms (including 0), together with an identity element. Let the elements of $S \cup \bar{S}$ act in the way corresponding to the multiplication in $(S)_{\text{reg}}$ (this will be described precisely). Then, take the semigroup $\langle S \cup \bar{S} \cup \{0\} \rangle_{F(Q \to Q)}$ generated in $F(Q \to Q)$ by the transformations $S \cup \bar{S} \cup \{0\}$.

We show that $\langle S \cup \bar{S} \cup \{0\} \rangle_F$ satisfies all the *axioms* (1)–(6) (this will be very tedious) and hence (by Fact 2.13) is a homomorphic image of $(S)_{\text{reg}}$. Thus we consider $(S)_{\text{reg}}$ as acting on Q (in a not necessarily faithful way).

Finally we show that elements of $(S)_{\text{reg}}$ which are represented by different coded normal forms act differently on Q. This shows that $(S)_{\text{reg}}$ acts faithfully on Q, and that different coded normal forms represent different elements of $(S)_{\text{reg}}$ – which proves uniqueness of coded normal forms.

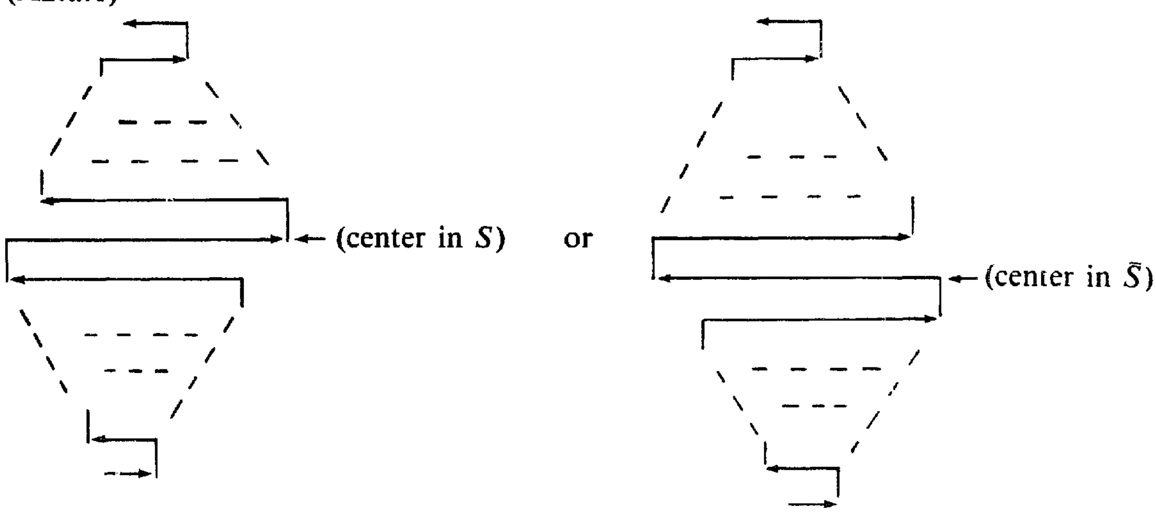
Remark. Lemma 2.26, which was used for Theorem 2.23, is an immediate corollary of the above proposition (and of the way a normal form is transformed into a coded normal form – see Fact A.1.2).

(a) *States and action*

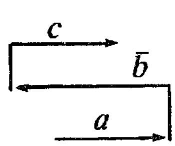
States. As just mentioned, we choose as state set Q the set of all *coded normal forms (including the zero), together with a new element I* (which will play the role of an identity or a start state). We still use a fixed set of representatives of L and R-classes of S.

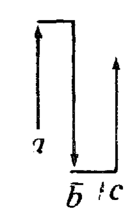
Graphical representation of the *states*: We still use the 'arrow picture' of Fact 2.2, but for notational reasons we draw the arrows horizontally, starting at the bottom. Upward pointing arrows now point to the right (forward direction) (see Section 2.2), downward pointing arrows now point to the left (backward direction). So states have the form

(A2.a.1)



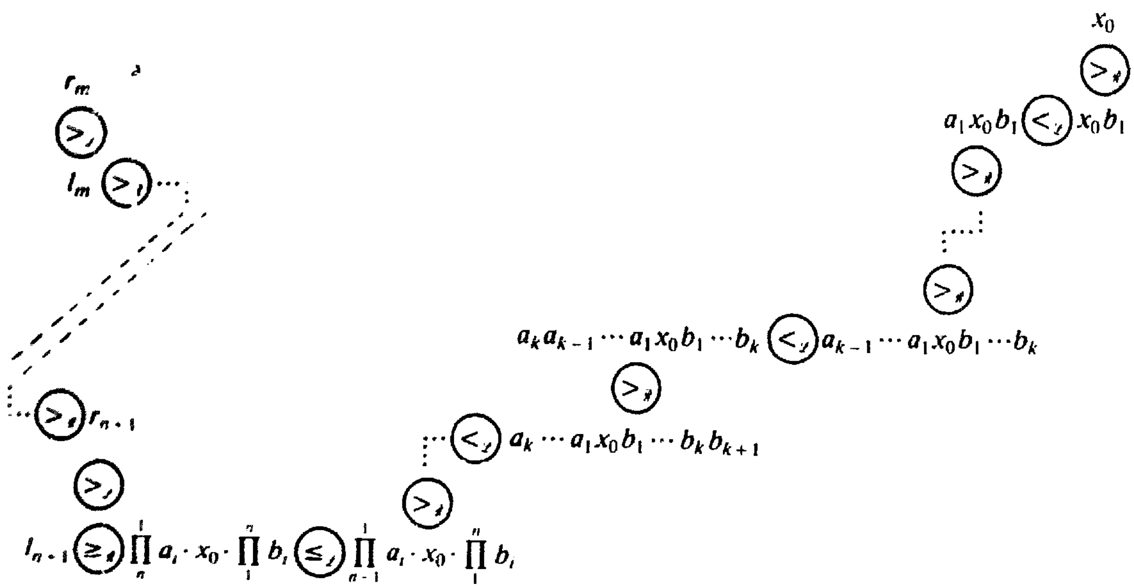
Example. The (coded) normal form $a\bar{b}c$ (with $a >_{\mathscr{L}} b <_{\mathscr{R}} c$) is represented as



(in Fact 2.2 it was drawn as ).

The reason why this notation is more convenient will appear when we define and study the action on the states.

(A2.a.2) A more explicit notation for states which we will use is



representing the coded normal form (with center $\in S$):

$$q = r_m \bar{l}_m \cdots \bar{l}_{n+1} c_n \bar{r}_n \cdots l_2 \bar{r}_2 l_1 \bar{r}_1 l_0$$

with

$$r_m >_{\mathscr L} l_m >_{\mathscr R} \cdots < l_{n+1} \geq_{\mathscr R} c_n \leq_{\mathscr L} r_n <_{\mathscr R} \cdots <_{\mathscr R} l_2 <_{\mathscr L} r_2 <_{\mathscr R} l_1 <_{\mathscr L} r_1 <_{\mathscr R} l_0$$

(the relations $\geq_{\mathscr R}$, $\leq_{\mathscr L}$ around c_n: not both $\equiv$)

and with

$$l_0 = x_0, \qquad r_1 = x_0 b_1, \qquad l_1 = a_1 x_0 b_1, \ldots,$$

$$l_{n-1} = \prod_{n-1}^{1} a_i \cdot x_0 \cdot \prod_{1}^{n-1} b_i, \qquad r_n = \prod_{n-1}^{1} a_i \cdot x_0 \cdot \prod_{1}^{n} b_i, \qquad c_n = \prod_{n}^{1} a_i \cdot x_0 \cdot \prod_{1}^{n} b_i.$$

Since we will define the actions on the right, only the right side of the state (coded normal form) will have to be written down in detail.

Action on the states. We shall define the action on Q of the elements $s \in S$, $\bar{s} \in \bar{S}$ and 0, and then take the semigroup generated by these actions in $F(Q \to Q)$. The action of s (resp. $\bar{s}, 0$) is denoted by (s) (resp. $(\bar{s})$, (0)).

First,

$$\forall q \in Q: \quad q \cdot (0) = 0.$$

And,

$$\forall s \in S: \quad I \cdot (s) = s, \quad I \cdot (\bar{s}) = \bar{s} \qquad (I \text{ is the identity}).$$

In general: if q is any coded normal form, then $q \cdot (s) = ((qs)\,norm)\,code$, where $(qs)\,norm$ denotes a particular *normal form* (described below) corresponding to the element $qs \in (S)_{\text{reg}}$; and $((qs)\,norm)\,code$ denotes the *coded* normal form obtained from the normal form $(qs)\,norm$ by using the procedure of A.1.2.

Remark. If $q = 0$, then $q(s) = 0$.

Similarly define $q \cdot (\bar{s}) = ((q\bar{s})\,norm)\,code$.

Normalization. The normalization $qs \to (qs)\,norm$ and $q\bar{s} \to (q\bar{s})\,norm$ will now be described more explicitly (where $s \in S$, q is a (coded) normal form).

Let $q = r_m \bar{l}_m \cdots r_{n+1} \bar{l}_{n+1} c_n \bar{r}_n \cdots l_2 \bar{r}_2 l_1 \bar{r}_1 l_0 \in Q$ be a *coded* normal form (with center $\in S$ for example; the case where center $\in \bar{S}$ is dual) with

$$l_0 = x_0, \qquad r_1 = x_0 b_1, \qquad l_1 = a_1 x_0 b_1, \ldots,$$

$$l_{k-1} = \prod_{k-1}^{1} a_i \cdot x_0 \cdot \prod_{1}^{k-1} b_i, \qquad r_k = \prod_{k-1}^{1} a_i \cdot x_0 \cdot \prod_{1}^{k} b_i, \ldots,$$

$$r_n = \prod_{n-1}^{1} a_i \cdot x_0 \cdot \prod_{1}^{n} b_i, \qquad c_n = \prod_{n}^{1} a_i \cdot x_0 \cdot \prod_{1}^{n} b_i.$$

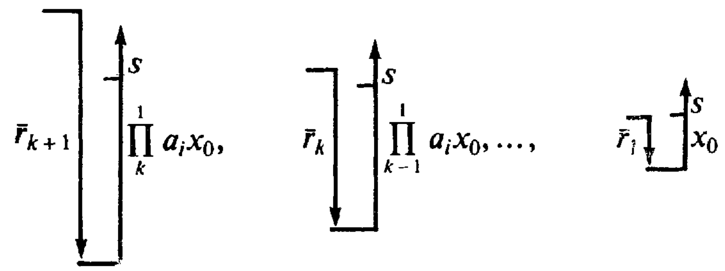
Then

$$(qs)\,norm = r_m \bar{l}_m \cdots \bar{l}_{n+1} c_n \bar{r}_n \cdots \bar{r}_{k+1} a_k \cdots a_1 x_0 s$$

if $r_{k+1} <_{\mathcal{R}} a_k \cdots a_1 x_0 s$, and $r_k \geq_{\mathcal{R}} a_{k-1} \cdots a_1 x_0 s, \ldots, r_1 \geq_{\mathcal{R}} x_0 s$,

arrow pictures of these conditions:

(A2.a.3)



$$(qs)\,norm = r_m \bar{l}_m \cdots \bar{l}_{n+1} a_n \cdots a_1 x_0 s \quad \text{if } r_n \geq_{\mathcal{R}} \prod_{n-1}^{1} a_i x_0 s, \ldots, r_1 \geq x_0 s,$$

$$(qs)\,norm = 0 \qquad \text{otherwise.}$$

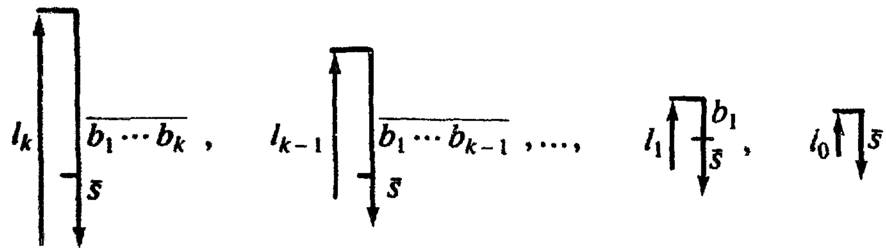
Hence $(qs)\,form$ is again a *normal form*, whose left side is coded but whose right side is not necessarily in coded form.

$$(q\bar{s})\,norm = r_m \bar{l}_m \cdots \bar{l}_{n+1} c_n \bar{r}_n \cdots l_1 \bar{r}_1 l_0 \bar{s} \quad \text{if } l_0 \leq_{\mathcal{L}} s,$$

$$(q\bar{s})\,norm = r_m \bar{l}_m \cdots \bar{l}_{n+1} c_n \bar{r}_n \cdots l_k \overline{s b_1 \cdots b_k} \quad \text{(where } n-1 \geq k \geq 1\text{)}$$

if $l_k <_{\mathcal{L}} s b_1 \cdots b_k$

and $l_{k-1} \geq_{\mathcal{L}} s b_1 \cdots b_{k-1}, \ldots, l_1 \geq_{\mathcal{L}} s b_1,\ l_0 \geq_{\mathcal{L}} s$, arrow picture of these conditions:



$(q\bar{s})\ norm = r_m \bar{l}_m \cdots r_{n+1} \bar{l}_{n+1} c_n s b_1 \cdots b_n$

if (as before) $l_{n-1} \geq_{\mathscr{L}} sb_1 \cdots b_{n-1}, \ldots, l_1 \geq_{\mathscr{L}} sb_1, l_0 \geq_{\mathscr{L}} s$,

and either $c_n <_{\mathscr{L}} sb_1 \cdots b_n$ or $l_{n+1} >_{\mathscr{R}} c_n \leq_{\mathscr{L}} sb_1 \cdots b_n$,

$(q\bar{s})\ norm = r_m \bar{l}_m \cdots r_{n+1} \bar{u} s b_1 \cdots b_n$

if $l_{n-1} \geq_{\mathscr{L}} sb_1 \cdots b_{n-1}, \ldots, l_1 \geq_{\mathscr{L}} sb_1, l_0 \geq_{\mathscr{L}} s$,

and $l_{n+1} \equiv_{\mathscr{R}} c_n \geq_{\mathscr{L}} sb_1 \cdots b_n$ (with $uc_n = sb_1 \cdots b_n$),

$(q\bar{s})\ norm = 0$ otherwise.

The case where the center of q is in $\bar{S}$ is dual to the one described.

Remark. If we consider $s, q, (qs)\ norm, qs$ as elements of $(S)_{\text{reg}}$, then it follows from the above description that $qs \cong (qs)\ norm$ in $(S)_{\text{reg}}$.

Graphical representation of the actions (if result $\neq 0$, and if $n > k$, i.e. $a_k \cdots a_1 x_0 s$ is not the center of the resulting state) is shown in Fig. 5.

Coding of the right side of a normal form. We mentioned that if q is a coded normal form, then $(qs)\ norm$ and $(q\bar{s})\ norm$ are again normal forms, whose left side is coded but whose right side is usually not in coded form. So, in order to obtain $q \cdot (s)$ and $q \cdot (\bar{s})$ we still have to describe explicitly how $(qs)\ norm$ and $(q\bar{s})\ norm$ are coded.

(A2.a.4) Let q be a normal form whose left side is coded. So

$$q = r_m \bar{l}_m \cdots \bar{l}_{n+1} x_n \bar{y}_n \cdots x_2 \bar{y}_2 x_1 \bar{y}_1 x_0$$

with

$$r_m >_{\mathscr{L}} l_m >_{\mathscr{R}} \cdots >_{\mathscr{L}} l_{n+1} \geq_{\mathscr{R}} x_n \leq_{\mathscr{L}} y_n <_{\mathscr{R}} \cdots <_{\mathscr{R}} x_2 <_{\mathscr{L}} y_2 <_{\mathscr{R}} x_1 <_{\mathscr{L}} y_1 <_{\mathscr{R}} x_0,$$

and $l_m, r_m, \ldots, r_{n+1}, l_{n+1}$ are among the chosen representatives of L (resp. R) -classes of S, and $y_1 = x_0 b_1$, $x_1 = a_1 x_0 b_1, \ldots, y_n = a_{n-1} \cdots a_1 x_0 b_1 \cdots b_n, x_n = a_n \cdots a_1 x_0 b_1 \cdots b_n$.

(Here we consider a normal q whose center is in S; the case where the center is in $\bar{S}$ is treated dually.) Then (see the coding procedure in A1.2):

$$(q)\ code = \left[\begin{array}{cc} & l_0 \\ & \textcircled{$>_{\mathscr{R}}$} \\ \left[\begin{array}{c} a_1 x_0 b_1 \textcircled{$<_{\mathscr{L}}$} \lambda_0 x_0 b_1 \\ \textcircled{$>_{\mathscr{R}}$} \\ \cdots \ \vdots \end{array} \right] & code \end{array} \right]$$

where $x_0 \equiv_{\mathscr{L}} l_0$ (= representative of L-class of x_0), and $\lambda_0 \in S^1$ *is such that* $\lambda_0 x_0 = l_0$.

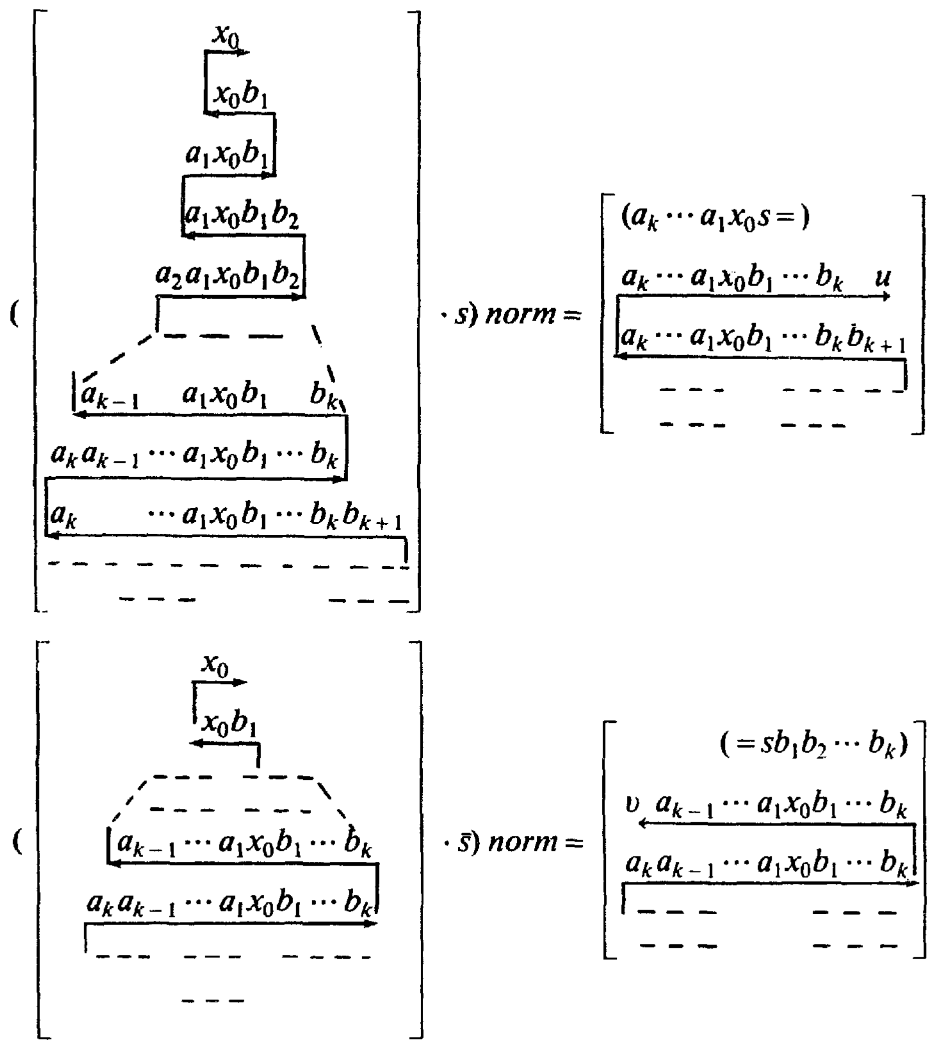


Fig. 5.

Remark. The element $\lambda_0x_0b_1$ does not depend on the choice of λ_0 such that $\lambda_0x_0=l_0$ (since $\lambda_0x_0b_1=l_0b_1$).

Claim 1. $x_0b_1\equiv_{\mathscr{L}}\lambda_0x_0b_1$.

Proof. $l_0=\lambda_0x_0\equiv_{\mathscr{L}}x_0\Rightarrow$(multiplying by b_1): $\lambda_0x_0b_1\equiv_{\mathscr{L}}x_0b_1$.

l_0

$\bigcirc\!\!\!>_{\mathscr{R}}$

Corollary 1′. $a_1x_0b_1 \bigcirc\!\!\!\!<_{\mathscr{L}} \lambda_0x_0b_1$ *holds.*

Proof. Obviously $\lambda_0x_0b_1=l_0b_1\leq_{\mathscr{R}}l_0$; if we had $\lambda_0x_0b_1\equiv_{\mathscr{R}}l_0$, then (since $\lambda_0x_0\equiv_{\mathscr{L}}x_0$ there exists λ_0^*: $\lambda_0^*\lambda_0x_0=x_0$) we would have $\lambda_0^*\lambda_0x_0b_1\equiv_{\mathscr{R}}\lambda_0^*\lambda_0x_0$, thus $x_0b_1\equiv_{\mathscr{R}}x_0$ – which contradicts $x_0b_1<_{\mathscr{R}}x_0$.

For the $\textcircled{<}_{\mathcal{L}}$-relation: we have $a_1x_0b_1 <_{\mathcal{L}} x_0b_1$, $x_0b_1 \equiv_{\mathcal{L}} \lambda_0x_0b_1$ (by the above claim). Hence $a_1x_0b_1 <_{\mathcal{L}} \lambda_0x_0b_1$. □

The codings can be continued inductively:

$$(q)\ code = \left[\begin{array}{cc} & \lambda_0x_0=l_0 \\ & \textcircled{>}_{\mathcal{R}} \\ \left[\begin{array}{c} a_1x_0b_1\varrho_1 \\ \textcircled{>}_{\mathcal{R}} \\ \vdots\cdots\textcircled{<}_{\mathcal{L}}\, a_1x_0b_1b_2 \end{array}\right] & \textcircled{<}_{\mathcal{L}}\, \lambda_0x_0b_1\varrho_1=r_1 \\ & code \end{array}\right],$$

where $\lambda_0x_0b_1 \equiv_{\mathcal{R}} r_1$ (= the representative of the R-class of $\lambda_0x_0b_1$), and $\varrho_1 \in S^1$ is such that $r_1=\lambda_0x_0b_1\varrho_1$.

Remark. The element $a_1x_0b_1\varrho_1$ does not depend on the choice of ϱ_1 such that $r_1=\lambda_0x_0b_1\varrho_1$ (Indeed, if $r_1=\lambda_0x_0b_1\varrho_1=\lambda_0x_0b_1\varrho_1'$, then multiplying by λ_0^* such that $\lambda_0^*\lambda_0x_0=x_0$: $\lambda_0^*r_1=x_0b_1\varrho_1=x_0b_1\varrho_1'$, hence $a_1x_0b_1\varrho_1=a_1x_0b_1\varrho_1'$).

Claim 2. $x_0b_1 \equiv_{\mathcal{R}} x_0b_1\varrho_1$. (Hence, multiplying by a_1: $a_1x_0b_1 \equiv_{\mathcal{R}} a_1x_0b_1\varrho_1$.)

Proof. We have $\lambda_0x_0b_1 \equiv_{\mathcal{R}} r_1=\lambda_0x_0b_1\varrho_1$. Multiply to the left by λ_0^* (such that $\lambda_0^*\lambda_0x_0=\lambda_0^*l_0=x_0$): $x_0b_1 \equiv_{\mathcal{R}} \lambda_0^*r_1=x_0b_1\varrho_1$.

Claim 2′. *The following strict relations hold:*

$$\left\{\begin{array}{ll} & l_0\ (=\lambda_0x_0) \\ & \textcircled{>}_{\mathcal{R}} \\ a_1x_0b_1\varrho_1 & \textcircled{<}_{\mathcal{L}}\, r_1\ (=\lambda_0x_0b_1\varrho_1) \\ \textcircled{>}_{\mathcal{R}} & \\ a_1x_0b_1b_2 & \end{array}\right\}$$

Proof. (a) We obviously have $\lambda_0x_0b_1\varrho_1 \le_{\mathcal{R}} \lambda_0x_0$. If we had $\lambda_0x_0b_1\varrho_1 \equiv_{\mathcal{R}} \lambda_0x_0$, then (multiplying by λ_0^* such that $\lambda_0^*\lambda_0x_0=x_0$): $x_0b_1\varrho_1 \equiv_{\mathcal{R}} x_0$. Hence, by the above claim: $x_0b_1 \equiv_{\mathcal{R}} x_0b_1\varrho_1 \equiv_{\mathcal{R}} x_0$; this contradicts the fact that $x_0b_1 <_{\mathcal{R}} x_0$, and proves $\lambda_0x_0b_1\varrho_1 <_{\mathcal{R}} \lambda_0x_0$.

(b) To show that $a_1x_0b_1\varrho_1 <_{\mathcal{R}} \lambda_0x_0b_1\varrho_1$, use Corollary 1′: $a_1x_0b_1 <_{\mathcal{L}} \lambda_0x_0b_1$, hence $a_1x_0b_1\varrho_1 \le_{\mathcal{L}} \lambda_0x_1b_1\varrho_1$. If we had $a_1x_0b_1\varrho_1 \equiv_{\mathcal{L}} \lambda_0x_1b_1\varrho_1$, multiply by ϱ_1^* such that (by Claim 2) $x_0b_1\varrho_1\varrho_1^*=x_0b_1$: then $a_1x_0b_1 \equiv_{\mathcal{L}} \lambda_0x_0b_1$, hence (by Claim 1: $x_0b_1 \equiv_{\mathcal{L}} \lambda_0x_0b_1$): $a_1x_0b_1 \equiv_{\mathcal{L}} x_0b_1$, which contradicts the fact that $a_1x_0b_1 <_{\mathcal{L}} x_0b_1$.

(c) To show that $a_1x_0b_1b_2 <_{\mathcal{R}} a_1x_0b_1\varrho_1$ use Claim 2, by which $a_1x_0b_1\varrho_1 \equiv_{\mathcal{R}} a_1x_0b_1$; and it is given that $a_1x_0b_1b_2 <_{\mathcal{R}} a_1x_0b_1$.

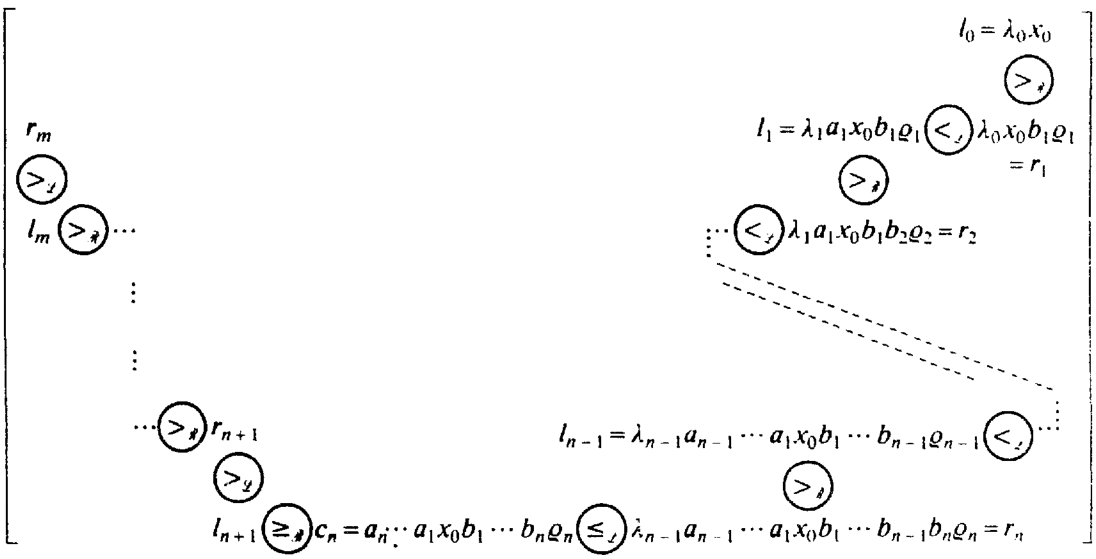
This *coding procedure* can be continued *inductively*: multipliers $\lambda_0, \lambda_1, \ldots, \lambda_{n-1}, \varrho_1, \ldots, \varrho_n$ are introduced such that

$$l_0 = \lambda_0 x_0 \quad (\equiv_{\mathcal{L}} x_0),$$
$$r_1 = \lambda_0 x_0 b_1 \varrho_1 \quad (\equiv_{\mathcal{R}} \lambda_0 x_0 b_1),$$
$$l_1 = \lambda_1 a_1 x_0 b_1 \varrho_1 \quad (\equiv_{\mathcal{L}} a_1 x_0 b_1 \varrho_1),$$
$$r_2 = \lambda_1 a_1 x_0 b_1 b_2 \varrho_2 \quad (\equiv_{\mathcal{R}} \lambda_1 a_1 x_0 b_1 b_2),$$
$$\ldots$$

(A2.a.5) $$l_{k-1} = \lambda_{k-1} a_{k-1} \cdots a_1 x_0 b_1 \cdots b_{k-1} \varrho_{k-1} \quad (\equiv_{\mathcal{L}} a_{k-1} \cdots a_1 x_0 b_1 \cdots b_{k-1} \varrho_{k-1}),$$
$$r_k = \lambda_{k-1} a_{k-1} \cdots a_1 x_0 b_1 \cdots b_{k-1} b_k \varrho_k \quad (\equiv_{\mathcal{R}} \lambda_{k-1} a_{k-1} \cdots a_1 x_0 b_1 \cdots b_{k-1} b_k),$$

for $1 \le k \le n$, where l_{k-1}, r_k are representatives of L (resp. R) -classes (in S); and $c_n = a_n \cdots a_1 x_0 b_1 \cdots b_n \varrho_n$.

(A2.a.6) **Fact.** *We have* (q) *code* =



We have to prove that this is indeed a normal form, i.e., that for $1 \le k < n$:

$$\begin{array}{l} l_{k-1} \\ \circled{>_{\mathcal{R}}} \\ l_k \circled{<_{\mathcal{L}}} r_k \end{array}, \quad \text{and} \quad c_n \circled{\le_{\mathcal{L}}} r_n.$$

For that we shall use the following claims:

(A2.a.7) Claim (λ).

$$\lambda_{k-1}a_{k-1}\cdots a_1x_0b_1\cdots b_{k-1}\equiv_{\mathscr{L}}a_{k-1}\cdots a_1x_0b_1\cdots b_{k-1}\quad \textit{for } 0<k<n.$$

Claim (ϱ).

$$a_{k-1}\cdots a_1x_0b_1\cdots b_{k-1}b_k\varrho_k\equiv_{\mathscr{R}}a_{k-1}\cdots a_1x_0b_1\cdots b_{k-1}b_k\quad \textit{for } 0<k\leq n.$$

Proof. Induction on k. (For $k=1$, see Claims 1 and 2 above.)

Claim λ. By definition of λ_{k-1} (see (A2.a.5)):

$$\lambda_{k-1}a_{k-1}\cdots x_0\cdots b_{k-1}\varrho_{k-1}\equiv_{\mathscr{L}}a_{k-1}\cdots x_0\cdots b_{k-1}\varrho_{k-1},$$

and (inductively, assuming Claim ϱ for $k-1$);

$$a_{k-2}\cdots x_0\cdots b_{k-1}\varrho_{k-1}\equiv_{\mathscr{R}}a_{k-2}\cdots x_0\cdots b_{k-1},$$

hence, there exists $\varrho^*_{k-1}\in S^1$ such that

$$a_{k-2}\cdots x_0\cdots b_{k-1}\varrho_{k-1}\varrho^*_{k-1}=a_{k-2}\cdots x_0\cdots b_{k-1}.$$

Now multiply the first line on the right by ϱ^*_{k-1}, and using the last line we obtain

$$\lambda_{k-1}a_{k-1}\cdots x_0\cdots b_{k-1}\equiv_{\mathscr{L}}a_{k-1}\cdots x_0\cdots b_{k-1}.$$

This proves Claim (λ).

Claim (ϱ). By the definition (A2.a.5) of ϱ_k;

$$\lambda_{k-1}a_{k-1}\cdots x_0\cdots b_k\varrho_k\equiv\lambda_{k-1}a_{k-1}\cdots x_0\cdots b_{k-1}.$$

And (by Claim λ for $k-1$, which has just been proved):

$$\lambda_{k-1}a_{k-1}\cdots x_0\cdots b_{k-1}\equiv_{\mathscr{L}}a_{k-1}\cdots x_0\cdots b_{k-1},$$

hence

$$(\exists\lambda^*_{k-1}\in S^1)\,\lambda^*_{k-1}\lambda_{k-1}a_{k-1}\cdots x_0\cdots b_{k-1}=a_{k-1}\cdots x_0\cdots b_{k-1}.$$

Multiplying the first line to the left by λ^*_{k-1} and using the last line:

$$a_{k-1}\cdots a_1x_0b_1\cdots b_k\varrho_k\equiv_{\mathscr{R}}a_{k-1}\cdots x_0\cdots b_k.$$

This proves Claim (ϱ). □

Proof that $l_k \textcircled{$<_{\mathscr{L}}$} r_k$. By Claim (λ):

$$r_k=\lambda_{k-1}a_{k-1}\cdots a_1x_0b_1\cdots b_{k-1}b_k\varrho_k\equiv_{\mathscr{L}}a_{k-1}\cdots a_1x_0b_1\cdots b_k\varrho_k,$$

moreover $l_k=\lambda_k a_k a_{k-1}\cdots a_1x_0b_1\cdots b_k\varrho_k$. Hence $l_k\leq_{\mathscr{L}}r_k$. We still must show $l_k\not\equiv_{\mathscr{L}}r_k$. If $l_k\equiv_{\mathscr{L}}r_k$, then

$$\lambda_k a_k a_{k-1}\cdots x_0\cdots b_k\varrho_k\equiv_{\mathscr{L}}a_{k-1}\cdots x_0\cdots b_k\varrho_k.$$

Multiplying on the right by ϱ^*_k satisfying $a_{k-1}\cdots\cdots b_k\varrho_k\varrho^*_k=a_{k-1}\cdots\cdots b_k$ (ϱ^*_k exists, since by Claim (ϱ), $a_{k-1}\cdots\cdots b_k\varrho_k\equiv_{\mathscr{R}}a_{k-1}\cdots\cdots b_k$) we obtain

$$\lambda_k a_k a_{k-1}\cdots x_0\cdots b_k\equiv_{\mathscr{L}}a_{k-1}\cdots x_0\cdots b_k.$$

And by Claim λ (for k):

$$\lambda_k a_k \cdots x_0 \cdots b_k \equiv_{\mathscr{L}} a_k \cdots x_0 \cdots b_k.$$

Combining the last two lines:

$$a_k a_{k-1} \cdots x_0 \cdots b_k \equiv_{\mathscr{L}} a_{k-1} \cdots x_0 \cdots b_k.$$

This however contradicts the fact that $a_k a_{k-1} \cdots x_0 \cdots b_k <_{\mathscr{L}} a_{k-1} \cdots x_0 \cdots b_k$ (which is given by the form of q).

Proof that $r_k <_{\mathscr{R}} l_{k-1}$. By Claim ($\lambda$):

$$r_k = \lambda_{k-1} a_{k-1} \cdots x_0 \cdots b_{k-1} b_k \varrho_k \quad (\equiv_{\mathscr{L}} a_{k-1} \cdots x_0 \cdots b_{k-1} b_k \varrho_k).$$

Moreover

$$l_{k-1} = \lambda_{k-1} a_{k-1} \cdots x_0 \cdots b_{k-1} \varrho_{k-1} \quad (\equiv_{\mathscr{L}} a_{k-1} \cdots x_0 \cdots b_{k-1} \varrho_{k-1} \text{ by Claim } \lambda).$$

By Claim ϱ (for $k-1$) there exists ϱ_{k-1}^* such that

$$a_{k-1} \cdots x_0 \cdots b_{k-1} \varrho_{k-1} \varrho_{k-1}^* = a_{k-2} \cdots x_0 \cdots b_{k-1}.$$

Hence

$$r_k = \lambda_{k-1} a_{k-1} a_{k-2} \cdots x_0 \cdots b_{k-1} \varrho_{k-1} \varrho_{k-1}^* b_k \varrho_k \leq_{\mathscr{R}} l_{k-1}.$$

We must still show $r_k \not\equiv_{\mathscr{R}} l_{k-1}$. Suppose we had $r_k \equiv_{\mathscr{R}} l_{k-1}$, i.e.,

$$\lambda_{k-1} a_{k-1} \cdots x_0 \cdots b_{k-1} b_k \varrho_k \equiv_{\mathscr{R}} \lambda_{k-1} a_{k-1} \cdots x_0 \cdots b_{k-1} \varrho_{k-1}$$

$\parallel\!\!|_{\mathscr{R}}$ (by Claim (ϱ), for k) $\parallel\!\!|_{\mathscr{R}}$ (by Claim (ϱ), for $k-1$)

hence

$$\lambda_{k-1} a_{k-1} \cdots x_0 \cdots b_{k-1} b_k \equiv_{\mathscr{R}} \lambda_{k-1} a_{k-1} \cdots x_0 \cdots b_{k-1}.$$

By Claim (λ) (for $k-1$) there exists λ_{k-1}^* such that

$$\lambda_{k-1}^* \lambda_{k-1} a_{k-1} \cdots x_0 \cdots b_{k-1} = a_{k-1} \cdots x_0 \cdots b_{k-1}.$$

Hence (multiplying by λ_{k-1}^*):

$$a_{k-1} \cdots x_0 \cdots b_{k-1} b_k \equiv_{\mathscr{R}} a_{k-1} \cdots x_0 \cdots b_{k-1}.$$

This however contradicts the strict $<_{\mathscr{R}}$ given by the form of q. □

This proves that (q) *code* has indeed the form indicated in Fact (A2.a.6).

We can write (q) *code* in terms of "a's and b's" (cf. notation of (A2.a.2)) as follows:

(A2.a.7′) By Claim λ (resp. ϱ) there exists λ_{k-1}^* (resp. ϱ_k^*) such that

for $0<k<n$: $\lambda_{k-1}^* \lambda_{k-1} a_{k-1} \cdots x_0 \cdots b_{k-1} = a_{k-1} \cdots x_0 \cdots b_{k-1}$,

and

for $0<k\leq n$: $a_{k-1} \cdots x_0 \cdots b_{k-1} b_k \varrho_k \varrho_k^* = a_{k-1} \cdots x_0 \cdots b_{k-1} b_k$.

Then denote $x_0^* = \lambda_0 x_0$, $b_1^* = b_1 \varrho_1$, and $a_{k-1}^* = \lambda_{k-1} a_{k-1} \lambda_{k-2}^*$ for $0<k<n$, $a_n^* = a_n \lambda_{n-1}^*$; and $b_k^* = \varrho_{k-1}^* b_k \varrho_k$ for $0<k\leq n$. Then (as is easy to check):

(A2.a.8) $\quad x_0^* = l_0, \qquad r_1 = x_0^* b_1^* \quad (=\lambda_0 x_0 b_1 \varrho_1);$

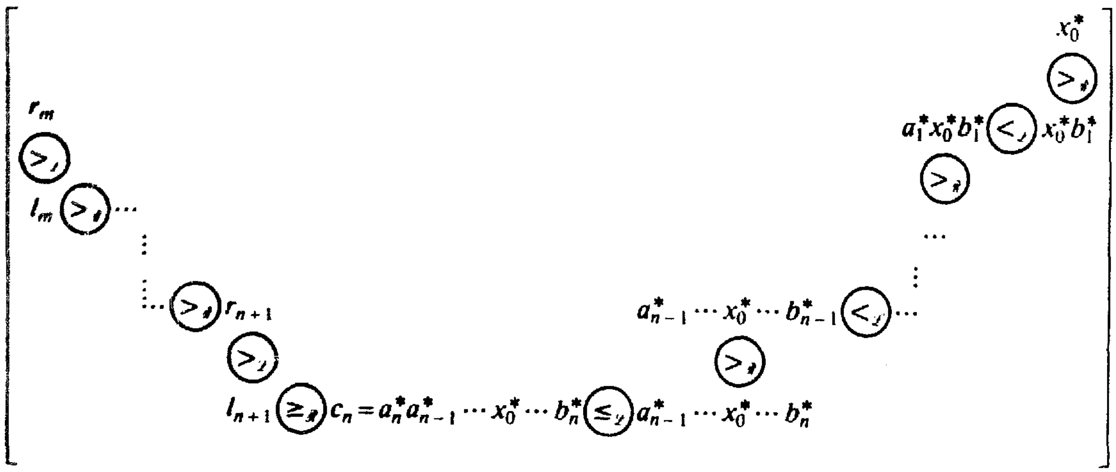
$$l_{k-1} = a_{k-1}^* \cdots a_1^* x_0^* b_1^* \cdots b_{k-1}^* \quad (=\lambda_{k-1} a_{k-1} \lambda_{k-2}^* \lambda_{k-2} a_{k-2} \lambda_{k-3}^* \cdots)$$

for $0<k<n$;

$$r_k = a_{k-1}^* \cdots a_1^* x_0^* b_1^* \cdots b_{k-1}^* b_k^* \quad \text{for } 0<k\leq n;$$

$$c_n = a_n^* \cdots a_1^* x_0^* b_1^* \cdots b_n^*.$$

Hence we have $(q)\,code =$



Remark. Compare this with the form given in (A2.a.4); see also (A.a.2). This completes the description of the *coding* of the right side of a normal form q.

Coding commutes with the actions. We shall now prove an important technical lemma.

(A2.a.9) **Lemma** ('Coding commutes with actions'). *Let q be a normal form whose left side is in coded form, and let $s \in S$, $\bar{s} \in \bar{S}$. Then*

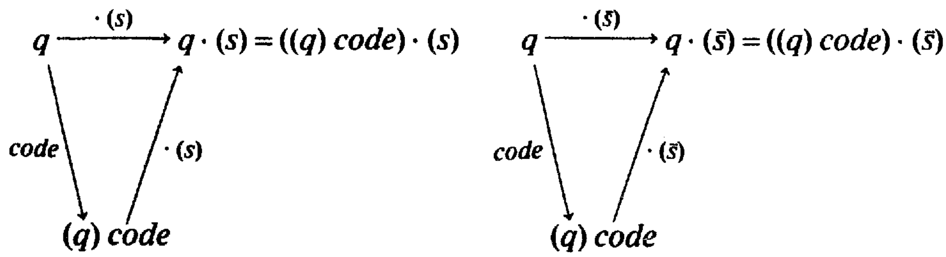
$$(qs)\,norm\,code = [((q)\,code)\,s]\,norm\,code,$$

or equivalently:

$$q \cdot (s) = ((q)\,code) \cdot (s)$$

(if we define $q \cdot (s) = (qs)\,norm\,code$); and $(q\bar{s})\,norm\,code = [((q)\,code)\,\bar{s}]\,norm\,code$, i.e., $q \cdot (\bar{s}) = ((q)\,code) \cdot (\bar{s})$).

Equivalently, the following diagrams commute:



Proof. We first consider the case of qs. We shall compute (qs) *norm* and $[(q)\,code)\,s]$ *norm*, and then show that a further coding makes both equal.

The normal form (qs) *norm* is described in (A2.a.3) – while $[(q)\,code\ s]$ *norm* is obtained by first replacing each a_i and b_j (and x_0) in q by a_i^*, resp. b_j^* and x_0^* (as described in (A2.a.8)), and then applying definition (A2.a.3) to this new form. Thus we need the following claim:

Claim

$$x_0^* s \leq_{\mathscr{R}} x_0^* b_1 \;\Leftrightarrow\; x_0 s \leq_{\mathscr{R}} x_0 b_1,$$

$$a_{k-1}^* \cdots a_1^* x_0^* s \leq_{\mathscr{R}} a_{k-1}^* \cdots a_1^* x_0^* b_1^* \cdots b_{k-1}^* b_k^* \;\Leftrightarrow$$

$$a_{k-1} \cdots a_1 x_0 s \leq_{\mathscr{R}} a_{k-1} \cdots a_1 x_0 b_1 \cdots b_{k-1} b_k, \quad \textit{for } 0<k\leq n,$$

$$a_{k-1}^* \cdots a_1^* x_0^* s >_{\mathscr{R}} a_{k-1}^* \cdots a_1^* x_0^* b_1^* \cdots b_k^* \;\Leftrightarrow$$

$$a_{k-1} \cdots a_1 x_0 s >_{\mathscr{R}} a_{k-1} \cdots a_1 x_0 b_1 \cdots b_k, \quad \textit{for } 0<k\leq n.$$

Proof of Claim. Since

$$r_k = a_{k-1}^* \cdots a_1^* x_0^* b_1^* \cdots b_k^* = \lambda_{k-1} a_{k-1} \cdots a_1 x_0 b_1 \cdots b_k \varrho_k$$

(by A2.a.5), and since (by Claim (ϱ) of A2.a.7):

$$a_{k-1} \cdots a_1 x_0 b_1 \cdots b_k \varrho_k \equiv_{\mathscr{R}} a_{k-1} \cdots a_1 x_0 b_1 \cdots b_k,$$

we have (multiplying the last relation on the left by λ_{k-1}):

$$r_k \equiv_{\mathscr{R}} \lambda_{k-1} a_{k-1} \cdots a_1 x_0 b_1 \cdots b_k.$$

So

$$a_{k-1}^* \cdots a_1^* x_0^* s \leq_{\mathscr{R}} a_{k-1}^* \cdots a_1^* x_0^* b_1^* \cdots b_k^* \;\Leftrightarrow$$

$$a_{k-1}^* \cdots a_1^* x_0^* s \leq_{\mathscr{R}} \lambda_{k-1} a_{k-1} \cdots a_1 x_0 b_1 \cdots b_k,$$

and

$$a_{k-1}^* \cdots a_1^* x_0^* s >_{\mathscr{R}} a_{k-1}^* \cdots a_1^* x_0^* b_1^* \cdots b_k^* \;\Leftrightarrow$$

$$a_{k-1}^* \cdots a_1^* x_0^* s >_{\mathscr{R}} \lambda_{k-1} a_{k-1} \cdots a_1 x_0 b_1 \cdots b_k, \quad \text{for } 0<k\leq n.$$

Moreover

$$a_{k-1}^* \cdots a_1^* x_0^* s = \lambda_{k-1} a_{k-1} \lambda_{k-2}^* \lambda_{k-2} a_{k-2} \lambda_{k-3}^* \cdots \lambda_1 a_1 \lambda_0^* \lambda_0 x_0 s$$
$$= \lambda_{k-1} a_{k-1} a_{k-2} \cdots a_1 x_0 s.$$

Thus

$$a_{k-1}^* \cdots a_1^* x_0^* s \le_{\mathscr{R}} a_{k-1}^* \cdots a_1^* x_0^* b_1^* \cdots b_k^* \Leftrightarrow$$

$$\lambda_{k-1} a_{k-1} \cdots a_1 x_0 s \le_{\mathscr{R}} \lambda_{k-1} a_{k-1} \cdots a_1 x_0 b_1 \cdots b_k;$$

the last relation implies (by multiplying on the left by λ_{k-1}^*):

$$a_{k-1} \cdots a_1 x_0 s \le_{\mathscr{R}} a_{k-1} \cdots a_1 x_0 b_1 \cdots b_k.$$

Also, this last relation implies (by multiplying now on the left by λ_{k-1}):

$$\lambda_{k-1} a_{k-1} \cdots a_1 x_0 s \le_{\mathscr{R}} \lambda_{k-1} a_{k-1} \cdots a_1 x_0 b_1 \cdots b_k.$$

Thus

$$a_{k-1}^* \cdots a_1^* x_0^* s \le_{\mathscr{R}} a_{k-1}^* \cdots a_1^* x_0^* b_1^* \cdots b_k^* \Leftrightarrow$$

$$a_{k-1} \cdots a_1 x_0 s \le_{\mathscr{R}} a_{k-1} \cdots a_1 x_0 b_1 \cdots b_k,$$

which is half of what was to be proved.

We still have to prove

$$a_{k-1}^* \cdots a_1^* x_0^* s >_{\mathscr{R}} a_{k-1}^* \cdots a_1^* x_0^* b_1^* \cdots b_k^* \Leftrightarrow$$

$$a_{k-1} \cdots a_1 x_0 s >_{\mathscr{R}} a_{k-1} \cdots a_1 x_0 b_1 \cdots b_k.$$

This will follow easily from

$$a_{k-1}^* \cdots a_1^* x_0^* s \ge_{\mathscr{R}} a_{k-1}^* \cdots a_1^* x_0^* b_1^* \cdots b_k^* \Leftrightarrow$$

$$a_{k-1} \cdots a_1 x_0 s \ge_{\mathscr{R}} a_{k-1} \cdots a_1 x_0 b_1 \cdots b_k,$$

and the above (since $>_{\mathscr{R}} \Leftrightarrow \ge_{\mathscr{R}} \& \not\le_{\mathscr{R}}$). This however is proved exactly like the above (where we had $\le_{\mathscr{R}}$ instead of $\ge_{\mathscr{R}}$). This proves the claim. □

From the above claim it follows that if q has the form given in (A2.a.4), then

$$(qs)\,norm = \left[\begin{array}{lcr}
r_m & \vdots & a_k \cdots a_1 x_0 s \\
\enclose{circle}{>_{\mathscr{L}}} & \vdots & \enclose{circle}{>_{\mathscr{R}}} \\
l_m \enclose{circle}{>_{\mathscr{R}}} \cdots & & \cdots \enclose{circle}{<_{\mathscr{L}}} a_k \cdots a_1 x_0 b_1 \cdots b_k b_{k+1} \\
 & \vdots & \\
 & \enclose{circle}{>_{\mathscr{R}}} & \\
 & \vdots \enclose{circle}{\ge_{\mathscr{R}}} c_n \enclose{circle}{\le_{\mathscr{L}}} a_{n-1} \cdots x_0 \cdots b_n & \\
 & \| & \\
 & a_n \cdots x_0 \cdots b_n &
\end{array}\right]$$

if and only if $((q)\,code\cdot s)\,norm=$

$$\begin{bmatrix} r_m & & & a_k^* \cdots a_1^* x_0^* s \\ \textcircled{$>_{\mathscr{L}}$} & & & \textcircled{$>_{\mathscr{R}}$} \\ l_m \textcircled{$>_{\mathscr{R}}$} \cdots & & & \cdots \textcircled{$<_{\mathscr{L}}$} a_k^* \cdots a_1^* x_0^* b_1^* \cdots b_{k-1}^* b_k^* \\ & & \vdots & \\ & & \textcircled{$>_{\mathscr{R}}$} & \\ & \vdots \textcircled{$\geq_{\mathscr{R}}$} c_n^* \textcircled{$\leq_{\mathscr{L}}$} a_{n-1}^* \cdots x_0^* \cdots b_n^* & & \\ & \| & & \\ & a_n^* \cdots x_0^* \cdots b_n^* & & \end{bmatrix}$$

Here we assume $n>k$ (i.e. $a_k\cdots a_1x_0s$ is not the center); the case $n=k$ is identical.

Finally, to prove Lemma (A2.a.9) we apply *code* to both forms. We have $a_k\cdots a_1x_0s\equiv_{\mathscr{L}} a_k^*\cdots a_1^*x_0^*s$ $(=\lambda_k a_k\cdots a_1x_0s)$. [This holds because $\lambda_k a_k\cdots a_1x_0s\leq_{\mathscr{L}} a_k\cdots a_1x_0s$ and because $\lambda_k^*\lambda_k a_k\cdots a_1x_0s=\lambda_k^*\lambda_k a_k\cdots a_1x_0b_1\cdots b_k u$ for some $u\in S^1$ (since $a_{k-1}\cdots a_1x_0s\leq_{\mathscr{R}} a_{k-1}\cdots a_1x_0b_1\cdots b_k$ by (A2.a.3)); $=a_k\cdots a_1x_0b_1\cdots b_k u$ (by the definition of λ_k^*, (A2.a.7')); $=a_k\cdots a_1x_0a$; hence $a_k\cdots a_1x_0s\leq_{\mathscr{L}}\lambda_k a_k\cdots a_1x_0s$.] (If $n=k$, then $a_k\cdots a_1x_0s=a_k^*\cdots a_1^*x_0^*s$.) Since $a_k^*\cdots a_1^*x_0^*s\equiv_{\mathscr{L}} a_k\cdots a_1x_0s$, both have the same L-class representative l_k'. Let λ_k' be such that $l_k'=\lambda_k'\lambda_k a_k\cdots a_1x_0s$, and let $\lambda_k''=\lambda_k'\lambda_k$; so $l_k'=\lambda_k''a_k\cdots a_1x_0s$. Hence (qs) *norm code*$=$

$$\begin{bmatrix} r_m & & l_k' & \\ \textcircled{$>_{\mathscr{L}}$} & & \textcircled{$>_{\mathscr{R}}$} & \\ l_m \textcircled{$>_{\mathscr{R}}$} \cdots\vdots & \begin{bmatrix} & \vdots\cdots \textcircled{$<_{\mathscr{L}}$} \lambda_k'' a_k \cdots a_1 x_0 b_1 \cdots b_k b_{k+1} \\ & \\ & \\ c_n \textcircled{$\leq_{\mathscr{L}}$} \cdots\vdots & \end{bmatrix} & & code \\ \cdots\vdots & & & \\ \textcircled{$>_{\mathscr{L}}$} & & & \\ l_{n+1} \textcircled{$\geq_{\mathscr{R}}$} & & & \end{bmatrix}$$

and $((q\,code)\,s)$ *norm code*$=$

$$\begin{bmatrix} r_m & & l_k' & \\ \textcircled{$>_{\mathscr{L}}$} & & \textcircled{$>_{\mathscr{R}}$} & \\ l_m \cdots\vdots & \begin{bmatrix} & \vdots\cdots \textcircled{$<_{\mathscr{L}}$} \lambda_k' a_k^* \cdots a_1^* x_0^* b_1^* \cdots b_k^* b_{k+1}^* \\ & \\ & \\ c_n^* \textcircled{$\leq_{\mathscr{L}}$} \cdots\vdots & \end{bmatrix} & & code \\ \vdots & & & \\ \textcircled{$>_{\mathscr{L}}$} & & & \\ l_{n+1} \textcircled{$\geq_{\mathscr{R}}$} & & & \end{bmatrix}$$

(by the description of the coding in (A2.a.4)). Moreover

$$\lambda_k''a_k\cdots a_1x_0b_1\cdots b_{k+1}=\lambda_k'\lambda_k a_k\cdots a_1x_0b_1\cdots b_{k+1}\qquad (\text{since } \lambda_k''=\lambda_k'\lambda_k),$$

and

$$\lambda_k'a_k^*\cdots a_1^*x_1^*b_1^*\cdots b_{k+1}^*=\lambda_k'\cdot\lambda_k a_k\cdots a_1x_0b_1\cdots b_{k+1}\varrho_{k+1}\quad (\text{by (A2.a.8)}).$$

But

$$\lambda_k'\lambda_k a_k\cdots a_1x_0b_1\cdots b_{k+1}\equiv_{\mathcal{R}}\lambda_k'\lambda_k a_k\cdots a_1x_0b_1\cdots b_{k+1}\varrho_{k+1}$$

(by Claim ϱ, (A2.a.7)); hence, continuing the coding, these two elements have the same R-class representative r_{k+1}'.

Let ϱ_{k+1}' be such that $r_{k+1}'=\lambda_k'\lambda_k a_k\cdots a_1x_0b_1\cdots b_{k+1}\varrho_{k+1}\varrho_{k+1}'$ (and let $\varrho_{k+1}''=\varrho_{k+1}\varrho_{k+1}'$). Hence (qs) *norm code* =

$$\left[\begin{array}{cccc} & & & l_k' \\ & & & \circledcirc(>_{\mathcal{R}}) \\ \vdots & \vdots & \left[\begin{array}{c} a_{k+1}\cdots x_0\cdots b_{k+1}\varrho_{k+1}'' \\ (>_{\mathcal{R}}) \\ \cdots \\ \vdots \\ c_n(\leq_{\mathcal{L}})\cdots \end{array}\right] & \begin{array}{c} (<_{\mathcal{L}})\,r_{k+1}' \\ \\ code \end{array} \\ & \vdots\cdots(\geq_{\mathcal{R}}) & & \end{array}\right],$$

and $((q\ code)\ s)$ *norm code* =

$$\left[\begin{array}{cccc} & & & l_k' \\ & & & (>_{\mathcal{R}}) \\ \vdots & \vdots & \left[\begin{array}{c} a_{k+1}^*\cdots x_0^*\cdots b_{k+1}^*\varrho_{k+1}' \\ (>_{\mathcal{R}}) \\ \cdots \\ \vdots \\ c_n^*(\leq_{\mathcal{L}})\cdots \end{array}\right] & \begin{array}{c} (<_{\mathcal{L}})\,r_{k+1}' \\ \\ code \end{array} \\ & \vdots\cdots(\geq_{\mathcal{R}}) & & \end{array}\right],$$

where

$$a_{k+1}\cdots x_0\cdots b_{k+1}\varrho_{k+1}''=a_{k+1}\cdots x_0\cdots b_{k+1}\varrho_{k+1}\varrho_{k+1}'\ (\text{since } \varrho_{k+1}''=\varrho_{k+1}\varrho_{k+1}'),$$

and

$$a_{k+1}^*\cdots x_0^*\cdots b_{k+1}^*\varrho_{k+1}'=\lambda_{k+1}a_{k+1}\cdots x_0\cdots b_{k+1}\varrho_{k+1}\cdot\varrho_{k+1}';$$

and by Claim (A2.a.7), these two are L-equivalent (thus will be coded to the same L-representative).

Continuing the coding inductively (which is easy now) we obtain Lemma (A2.a.9) for the case qs (the induction works out in the end, for c_n and c_n^*, since $\lambda_n=1\in S^1$).

The case of $q\bar{s}$ is very similar, and the case where the center of q is in $\bar{S}$ is dual. This proves (A2.a.9). □

We are now ready to verify that all the Axioms (1)–(6) hold for $\langle S\cup\bar{S}\cup\{0\}\rangle_{F(Q)}$.

(b) *Verification of the axioms* (for $\langle S\cup\bar{S}\cup\{0\}\rangle_{F(Q)}$)

Axioms (3) *and* (4) (which state that (0) acts like a zero) hold trivially since $\forall q\in Q$: $q\cdot(0)=0$.

Axiom (5). We first show that for all $q\in Q$, $s\in S$: $q(s)=q(s)\cdot(\bar{s})\cdot(s)$. If $q\cdot(s)=0$, then this certainly holds. So assume $q\cdot(s)\neq 0$. Suppose q and (qs) *norm* have the form indicated in (A2.a.3); so

$$q\cdot(s)=\left[\begin{array}{cccc} & & & a_k\cdots x_0 s\\ & & & >_{\mathscr{R}}\\ \vdots & & a_{k+1}a_k\cdots x_0\cdots b_{k+1} <_{\mathscr{L}} & a_k\cdots x_0\cdots b_{k+1}\\ & \vdots & >_{\mathscr{R}} & \\ & & \cdots & \\ \vdots\ \cdots\ \vdots & & & \end{array}\right] code$$

(assuming $k<n$; the reasoning is similar if $a_{k=n}\cdots a_1x_0s$ is the new center).

Next, let us compute $q(s)(\bar{s})(s)$: By Lemma (A2.a.9),

$$q(s)(\bar{s})(s)=((qs)\ norm\cdot\bar{s})\ norm\cdot s)\ norm\ code$$

(coding done only once, at the end).

If (1) $a_k\cdots a_1x_0s<_{\mathscr{L}}s$, or: if (2) $k=n$ (i.e., $a_n\cdots a_1x_0s$ is the center of (qs) *norm*) and $l_{n+1}\geq_{\mathscr{R}}a_n\cdots a_1x_0s$, then

$$q(s)(\bar{s})(s)=\begin{cases}\left[\left[\left[\begin{array}{cc} & a_k\cdots a_1x_0s <_{\mathscr{L}} s\\ & >_{\mathscr{R}}\\ \vdots & \cdots <_{\mathscr{L}} a_k\cdots x_0\cdots b_kb_{k+1}\\ \cdots & \end{array}\right]\cdot s\right]\right] norm\ code & \text{in case (1),}\\ \left[\left[\left[\begin{array}{cc} \vdots\ \vdots\ \vdots & \\ & \cdots >_{\mathscr{R}} a_n\cdots a_1x_0s \leq_{\mathscr{L}} s\end{array}\right]\cdot s\right]\right] norm\ code & \text{in case (2).}\end{cases}$$

In both cases

$$(((qs)\ norm\ \bar{s})\ norm\ s)\ norm=(qs)\ norm$$

(to compute $(((qs)\,norm\,\bar{s})\,norm\,\bar{s})\,norm$ from the above $((qs)\,norm\,\bar{s})\,norm$, use (A2.a.3), where x_0 is dropped, k is replaced by 1, $\bar{r}_1$ by $\bar{s}$, and l_1 by $a_k \cdots a_1 x_0 s$ etc.). Hence $q(s) = q(s)(\bar{s})(s)$ in these cases.

If (3) $a_k \cdots a_1 x_0 s \equiv_{\mathscr{L}} s$, $n > k$, then let b'_{k+1} be such that $a_k \cdots a_1 x_0 s b'_{k+1} = a_k \cdots a_1 x_0 b_1 \cdots b_k b_{k+1}$ (since $a_k \cdots a_1 x_0 s >_{\mathscr{R}} a_k \cdots a_1 x_0 b_1 \cdots b_{k+1}$); then, since $a_{k+1} a_k \cdots x_0 \cdots b_{k+1} <_{\mathscr{L}} a_k \cdots x_0 \cdots b_{k+1}$ we have $a_{k+1}(a_k \cdots x_0 s) b'_{k+1} <_{\mathscr{L}} (a_k \cdots x_0 s) b'_{k+1}$. Therefore by (A2.a.3):

$$((qs)\,norm\,\bar{s})\,norm \cdot s =$$

$$= \begin{bmatrix} & & & & a_k \cdots x_0 s \\ & & & & \textcircled{>_{\mathscr{R}}} \\ & & a_{k+1}(a_k \cdots x_0 s) b'_{k+1} & \textcircled{<_{\mathscr{L}}} & (a_k \cdots x_0 s) b'_{k+1} \\ \vdots & \vdots & \textcircled{>_{\mathscr{R}}} & & \\ & \cdots & \vdots & & \end{bmatrix} code \cdot (\bar{s}) \cdot (s)$$

$$= \left[\begin{bmatrix} & & a_{k+1}(a_k \cdots x_0 s) b'_{k+1} & \textcircled{<_{\mathscr{L}}} & s b'_{k+1} \\ \vdots & \vdots & \textcircled{>_{\mathscr{R}}} & & \\ & & \cdots & & \\ & \cdots & & & \end{bmatrix} \cdot s \right] norm\ code.$$

Moreover $s b'_{k+1} <_{\mathscr{R}} s$ (since *if* $s b'_{k+1} \equiv_{\mathscr{R}} s$, then $a_k \cdots x_0 s b'_{k+1} \equiv_{\mathscr{R}} a_k \cdots x_0 s$ which contradicts the fact that $a_k \cdots x_0 s b'_{k+1} <_{\mathscr{R}} a_k \cdots x_0 s$). Hence

$$q(s)(\bar{s})(s) = (qs)\,norm\,\bar{s}\,norm\,s\,norm\,code$$

$$= \begin{bmatrix} & & & & s \\ & & & & \textcircled{>_{\mathscr{R}}} \\ & & a_{k+1}(a_k \cdots x_0 s) b'_{k+1} & \textcircled{<_{\mathscr{L}}} & s b'_{k+1} \\ \vdots & \vdots & \textcircled{>_{\mathscr{R}}} & & \\ & \cdots & \vdots \quad \cdots & & \end{bmatrix} code$$

$$= \begin{bmatrix} & & & & s \\ & & & & \textcircled{>_{\mathscr{R}}} \\ & & a_{k+1} a_k \cdots x_0 b_1 \cdots b_k b_{k+1} & \textcircled{<_{\mathscr{L}}} & s b'_{k+1} \\ \vdots & \vdots & \textcircled{>_{\mathscr{R}}} & & \\ & \cdots & \vdots \quad \cdots & & \end{bmatrix} code.$$

However, by Fact 2.5, or by the proof of Fact (A1.2), this is equal to

$$\begin{bmatrix} & & & & a_k \cdots a_1 x_0 s \\ & & & & \textcircled{$>_{\mathcal{R}}$} \\ & & a_{k+1} \cdots a_1 x_0 b_1 \cdots b_{k+1} & \textcircled{$<_{\mathcal{L}}$} & a_k \cdots a_1 x_0 s b'_{k+1} \\ \vdots & \vdots & \textcircled{$>_{\mathcal{R}}$} & & \\ & \cdots \ \vdots & \cdots & & \end{bmatrix} \mathit{code}$$

since $a_k \cdots a_1 x_0 s \equiv_{\mathcal{L}} s$. Hence (since $a_k \cdots a_1 x_0 s b'_{k+1} = a_k \cdots a_1 x_0 b_1 \cdots b_k b_{k+1}$), we have $q \cdot (s) = q \cdot (s)(\bar{s})(s)$.

If (4) $n = k$ (i.e. $a_n \cdots a_1 x_0 s \equiv_{\mathcal{L}} s$), and $l_{n+1} \equiv_{\mathcal{R}} a_n \cdots a_1 x_0 s$ (the case $l_{n+1} >_{\mathcal{R}} \cdots$ was considered in (2)), then let b'_{n+1} be such that $l_{n+1} = a_n \cdots a_1 x_0 s b'_{n+1}$.

Claim. $s b'_{n+1} \equiv_{\mathcal{R}} s$.

Proof.

$$a_n \cdots a_1 x_0 s \equiv_{\mathcal{L}} s \ \Rightarrow \ (\exists u \in S^1)\, u a_n x_0 s = s.$$

Hence multiplying $(l_{n+1} =)\, a_n \cdots a_1 x_0 s b'_{n+1} \equiv_{\mathcal{R}} a_n \cdots a_1 x_0 s$ on the left by u we obtain $u a_n \cdots a_1 x_0 s b'_{n+1} \equiv_{\mathcal{R}} u a_n \cdots a_1 x_0 s$, which is equivalent to $s b'_{n+1} \equiv_{\mathcal{R}} s$. This proves the claim.

Now

$$q \cdot (s)(\bar{s})(s) = (qs)\ \mathit{norm\ code} \cdot (\bar{s})(s) =$$

$$= \begin{bmatrix} \vdots \ \cdots & & \\ \textcircled{$>_{\mathcal{L}}$} & & \\ l_{n+1} & \textcircled{$\equiv_{\mathcal{R}}$} & a_n \cdots a_1 x_0 s \\ \| & & \\ a_n \cdots a_1 x_0 s b'_{n+1} & & \end{bmatrix} \mathit{code} \cdot (\bar{s})(s)$$

$$= \begin{bmatrix} \vdots \ \cdots & & \\ \textcircled{$>_{\mathcal{L}}$} & & \\ s b'_{n+1} & \textcircled{$\equiv_{\mathcal{R}}$} & s \end{bmatrix} \cdot (\bar{s})(s) \qquad \text{(by the properties of } \mathit{code}\text{)},$$

$$= \begin{bmatrix} \vdots \ \cdots \\ \textcircled{$>_{\mathcal{L}}$} \\ s b'_{n+1} \end{bmatrix} \cdot (s) \qquad \text{(by definition of } (\bar{s})\text{)},$$

$$= \begin{bmatrix} \vdots \ \cdots & & \\ \textcircled{$>_{\mathcal{L}}$} & & \\ s b'_{n+1} & \textcircled{$\equiv_{\mathcal{R}}$} & s \end{bmatrix} \mathit{code}$$

$$= \begin{bmatrix} \vdots & \cdots & & \\ & \textcircled{>_{\mathscr{L}}} & & \\ a_n \cdots a_1 x_0 s b'_{n+1} \,(=l_{n+1}) & \textcircled{\equiv_{\mathscr{R}}} & a_n \cdots a_1 x_0 s \end{bmatrix} \textit{code},$$

which is equal to $q \cdot (s)$ in this case.

Here we assumed that q has its center in S; the case where the center of q is in $\bar{S}$ is treated identically.

This proves that the axiom $s = s\bar{s}s$ is satisfied.

The axiom $\bar{s} = \bar{s}s\bar{s}$ is verified in a similar manner.

Axiom (1). We have to show: $(\forall q \in Q)\, q \cdot (s_1)(s_2) = q \cdot (s_1 s_2)$.

It is easy to see that if $q(s_1)(s_2) \neq 0$ and $q(s_1 s_2) \neq 0$, then $q \cdot (s_1)(s_2) = q \cdot (s_1 s_2)$. We still have to show: $q \cdot (s_1)(s_2) \neq 0$ iff $q \cdot (s_1 s_2) \neq 0$.

Claim. $q \cdot (s_1)(s_2) \neq 0 \Rightarrow q \cdot (s_1 s_2) \neq 0$.

If $q \cdot (s_1)(s_2) \neq 0$, then $q \cdot (s_1) \neq 0$, hence by (A2.a.3), there exists h such that for all j with $0 \le j < h$:

$$\begin{cases} a_j \cdots a_1 x_0 s_1 \le_{\mathscr{R}} a_j \cdots a_1 x_0 b_1 \cdots b_j b_{j+1}, \\ a_h \cdots a_1 x_0 s_1 >_{\mathscr{R}} a_h \cdots a_1 x_0 b_1 \cdots b_h b_{h+1}. \end{cases}$$

Remark. Here we assume that q has its center in S, and that $h, k < n$. The other cases are similar.

And, since $(q \cdot (s_1))(s_2) \neq 0$, there exists k $(\ge h)$ such that for all j with $h \le j < k$:

$$\begin{cases} a_j \cdots a_h \cdots a_1 x_0 s_1 s_2 \le_{\mathscr{R}} a_j \cdots a_h \cdots a_1 x_0 b_1 \cdots b_h \cdots b_j b_{j+1}, \\ a_k \cdots a_h \cdots a_1 x_0 s_1 s_2 >_{\mathscr{R}} a_k \cdots a_h \cdots a_1 x_0 b_1 \cdots b_h \cdots b_k b_{k+1}. \end{cases}$$

(Here we use Lemma (A2.a.9).) But, since $a_j \cdots a_1 x_0 s_1 s_2 \le_{\mathscr{R}} a_j \cdots a_1 x_0 s_1$, the first conditions (for $q \cdot (s_1) \neq 0$) imply that for all j with $0 \le j < h$:

$$a_j \cdots a_1 x_0 s_1 s_2 \le_{\mathscr{R}} a_j \cdots a_1 x_0 b_1 \cdots b_j b_{j+1};$$

therefore $q \cdot (s_1 s_2) \neq 0$ (and these expressions also show that $q \cdot (s_1 s_2) = q \cdot (s_1)(s_2)$).

Claim. $q \cdot (s_1 s_2) \neq 0 \Rightarrow q \cdot (s_1)(s_2) \neq 0$. If $q \cdot (s_1 s_2) \neq 0$, then (by (A2.a.3)): there exists k such that for all j with $0 \le j < k$:

$$\begin{cases} a_j \cdots a_1 x_0 s_1 s_2 \le_{\mathscr{R}} a_j \cdots a_1 x_0 b_1 \cdots b_j b_{j+1}, \\ a_k \cdots a_1 x_0 s_1 s_2 >_{\mathscr{R}} a_k \cdots a_1 x_0 b_1 \cdots b_k b_{k+1}. \end{cases}$$

This implies that $(q \cdot (s_1))(s_2) \neq 0$, provided that $q \cdot (s_1) \neq 0$.

To show that $q \cdot (s_1) \neq 0$ we need the existence of a number h $(\le h)$ such that for all j with $0 \le j < h$:

$$\begin{cases} a_j \cdots a_1 x_0 s_1 \le_{\mathscr{R}} a_j \cdots a_1 x_0 b_1 \cdots b_j b_{j+1}, \\ a_h \cdots a_1 x_0 s_1 >_{\mathscr{R}} a_h \cdots a_1 x_0 b_1 \cdots b_h b_{h+1}. \end{cases}$$

However, since $(\forall j, 0 \le j < k)$:

$$\begin{cases} a_j \cdots a_1 x_0 s_1 s_2 \le_{\mathscr{R}} a_j \cdots a_1 x_0 s_1, \\ a_j \cdots a_1 x_0 s_1 s_2 \le_{\mathscr{R}} a_j \cdots a_1 x_0 b_1 \cdots b_j b_{j+1} \end{cases}$$

we have, by *unambiguity of the R-order* of S:

$$(\forall j, 0 \le j < k): \quad a_j \cdots a_1 x_0 s_1 \ \ge_{\mathscr{R}} \text{ or } <_{\mathscr{R}} \ a_j \cdots a_1 x_0 b_1 \cdots b_j b_{j+1}$$

Hence, there *exists*

$$h = \min\{ j \mid 0 \le j < k \text{ and } a_j \cdots a_1 x_0 s_1 >_{\mathscr{R}} a_j \cdots a_1 x_0 b_1 \cdots b_j b_{j+1} \}.$$

Now $q \cdot (s_1) \ne 0$, since for this choice of h we have $(\forall j, 0 j < h)$:

$$\begin{cases} a_j \cdots a_1 x_0 s_1 \le_{\mathscr{R}} a_j \cdots a_1 x_0 b_1 \cdots b_j b_{j+1}, \\ a_h \cdots a_1 x_0 s_1 >_{\mathscr{R}} a_h \cdots a_1 x_0 b_1 \cdots b_h b_{h+1}. \end{cases} \qquad \square$$

Axiom (2). That $q \cdot (\bar{s}_1)(\bar{s}_2) = q \cdot (s_2 s_1)$ is proved in a similar way; here we need the assumption that the *L-order of S is unambiguous*.

Axiom (6L). We shall show $(s_1)(\bar{s}_2) \ne 0$ iff $s_1 \gtreqless_{\mathscr{L}} s_2$.

($\Rightarrow$) Suppose for some $q \in Q$, we have $q \cdot (s_1)(\bar{s}_2) \ne 0$. Then $q \cdot (s_1)$ is of the form

$$\left[\begin{array}{ll} & a_k \cdots a_1 x_0 s_1 \\ & \quad \textcircled{$>_{\mathscr{R}}$} \\ \vdots & \cdots \textcircled{$<_{\mathscr{L}}$} a_k \cdots a_1 x_0 b_1 \cdots b_k b_{k+1} \\ \quad \vdots\ldots\vdots & \end{array} \right] \text{code,}$$

and $q \cdot (s_1)(\bar{s}_2)$ has one of the following two forms (using Lemma (A2.a.9)):

Case 1:

$$q \cdot (s_1)(\bar{s}_2) = \left[\begin{array}{ll} & a_k \cdots a_1 x_0 s_1 \textcircled{$<_{\mathscr{L}}$} s_2 \\ \vdots & \quad \textcircled{$>_{\mathscr{R}}$} \\ \quad \vdots & \cdots \textcircled{$<_{\mathscr{L}}$} a_k \cdots a_1 x_0 b_1 \cdots b_{k+1} \\ \quad \ldots \ \vdots & \end{array} \right] \begin{array}{l} \textit{code}, \\ \\ \text{if } a_k \cdots a_1 x_0 s_1 <_{\mathscr{L}} s_2. \end{array}$$

Case 2:

$$q \cdot (s_1)(\bar{s}_2) = \left[\begin{array}{l} \quad a_{k+p} \cdots a_{k+1} (a_k \cdots a_1 x_0 s_1) b'_{k+1} b_{k+2} \cdots b_{k+p} \\ = a_{k+p} \cdots a_1 x_0 b_1 \cdots b_{k+1} b_{k+2} \cdots b_{k+p} \textcircled{$<_{\mathscr{L}}$} s_2 b'_{k+1} b_{k+2} \cdots b_{k+p} \\ \quad \textcircled{$>_{\mathscr{R}}$} \\ \quad \cdots \end{array} \right] \textit{code}$$

(where b'_{k+1} is such that $a_k \cdots a_1 x_0 b_1 \cdots b_k b_{k+1} = a_k \cdots a_1 x_0 s_1 b'_{k+1} <_{\mathscr{R}} a_k \cdots a_1 x_0 s_1$), if

$$s_2 \leq_{\mathscr{L}} a_k \cdots a_1 x_0 s_1, \qquad s_2 b'_{k+1} \leq_{\mathscr{L}} a_{k+1}(a_k \cdots a_1 x_0 s_1) b'_{k+1}, \ldots$$

$$\ldots, s_2 b'_{k+1} b_{k+2} \cdots b_{k+p-1} \leq_{\mathscr{L}} a_{k+p-1} \cdots a_{k+1}(a_k \cdots a_1 x_0 s_1) b'_{k+1} b_{k+2} \cdots b_{k+p-1},$$

and

$$s_2 b'_{k+1} b_{k+2} \cdots b_{k+p} >_{\mathscr{L}} a_{k+p} \cdots a_{k+1}(a_k \cdots a_1 x_0 s_1) b'_{k+1} b_{k+2} \cdots b_{k+p}$$

$$(= a_{k+p} \cdots a_{k+1} a_k \cdots a_1 x_0 b_1 \cdots b_{k+1} b_{k+2} \cdots b_{k+p}).$$

Remark. Here we assume q has its center in S, and that $k<n$, $k+p<n$. The other cases are similar.

In *Case* 1 we have $a_k \cdots a_1 x_0 s_1 <_{\mathscr{L}} s_2$, and of course $a_k \cdots a_1 x_0 s_1 \leq_{\mathscr{L}} s_1$. Hence, by *unambiguity of the L-order* of S: $s_1 \gtreqless_{\mathscr{L}} s_2$.

In *Case* 2 we have (see above) $s_2 \leq_{\mathscr{L}} a_k \cdots a_1 x_0 s_1$, and of course $a_k \cdots a_1 x_0 s_1 \leq_{\mathscr{L}} s_1$; hence $s_2 \leq_{\mathscr{L}} s_1$.

So we proved that $(\exists q \in Q)\, q \cdot (s_1)(\bar{s}_2) \neq 0 \Rightarrow s_1 \gtreqless_{\mathscr{L}} s_2$.

($\Leftarrow$) From the definitions

$$I \cdot (s_1)(\bar{s}_2) = s_1 \cdot (\bar{s}_2) = \begin{cases} (s_1 \circledcirc{\leq_{\mathscr{L}}} s_2) \text{ code} & \text{if } s_1 \leq_{\mathscr{L}} s_2, \\ \begin{pmatrix} s_1 \\ \circledcirc{>_{\mathscr{L}}} \\ s_2 \end{pmatrix} \text{ code} & \text{if } s_1 >_{\mathscr{L}} s_2, \\ 0 & \text{if } s_1 \not\gtreqless_{\mathscr{L}} s_2. \end{cases}$$

Thus $(s_1)(\bar{s}_2) \neq 0$ if $s_1 \not\gtreqless_{\mathscr{L}} s_2$. □

The verification of Axiom (6R) is dual.

This proves that $\langle S \cup \bar{S} \cup \{0\} \rangle_{F(Q)}$ satisfies all the axioms, and thus is a homomorphic image of $(S)_{\text{reg}}$ (following the reasoning given at the beginning of A2); this also implies that $(S)_{\text{reg}}$ acts on Q.

We shall show next that elements of $(S)_{\text{reg}}$ which are represented by different coded normal forms act differently on Q; this implies that different coded normal forms represent different elements of $(S)_{\text{reg}}$ and that the action of $(S)_{\text{reg}}$ on Q is faithful (hence $(Q, (S)_{\text{reg}})$ is a transformation semigroup).

(c) *Uniqueness*

Uniqueness follows from the following fact. Let $r_m \bar{l}_m \cdots \cdots l_1 \bar{r}_1 l_0$ be a coded normal form (with center in S or in $\bar{S}$). Then

$$I \cdot (r_m)(\bar{l}_m) \cdots \cdots (l_1)(\bar{r}_1)(l_0) = r_m \bar{l}_m \cdots \cdots l_1 r_1 l_0.$$

(This can be proved easily from the definitions of *norm* and *code*.)

Also $I \cdot (0) = 0$. This completes the proof. □

Acknowledgement

I thank Prof. John Rhodes for his encouragement and for the stimulating discussions I had with him while I was working on this problem (which is part of my Ph.D. thesis).

References

[1] J.C. Birget, Iteration of expansions – Unambiguous semigroups, J. Pure Appl. Algebra, in this issue.
[2] J.C. Birget and J. Rhodes, Almost finite expansions of arbitrary semigroups, J. Pure Appl. Algebra 32 (1984) 239–287.
[3] J.C. Birget, The synthesis theorem for regular semigroups and its generalization to arbitrary semigroups (in preparation).
[4] S. Lazarus and J. Rhodes, Infinite holonomy decomposition (in preparation).
[5] A.H. Clifford and G.B. Preston, The Algebraic Theory of Semigroups, Vols. 1, 2, Mathematical Surveys 7 (Amer. Math. Soc., Providence, RI, 1961, 1967).
[6] J.M. Howie, An Introduction to Semigroup Theory (Academic Press, New York, 1976).
[7] G. Lallement, Semigroups and Combinatorial Applications (Wiley, New York, 1979).
[8] J. Rhodes, Infinite iteration of matrix semigroups, Parts I & II, submitted to J. Algebra.
[9] J. Rhodes and D. Allen, Synthesis of classical and modern semigroup theory, Advances in Math. 11 (1976) 238–266.